California State University, San Marcos

Approval Sheet

Thesis Submitted for Partial Fulfillment
of the Requirements for the Degree

Masters of Science
in
Mathematical Sciences

Thesis title: Unimodal Dynamical Systems and Chaos

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12/22/97

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Unimodal Dynamical Systems and Chaos

by

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1 ~ Introduction

This paper will look at some properties of dynamical systems. We will begin with basic definitions and concepts of this subject, then we will focus on a broad group of one-dimensional systems: unimodal functions.

Our main interest will be in looking at how data iterates through these functions. The set of all values that a given initial value takes on as it iterates through a function is called that value’s orbit. This paper’s major concern is looking at the formation and behavior of these orbits with particular emphasis on periodic orbits. Periodic orbits are those orbits that endlessly repeat the same set of values.

We will look how the existence of certain periodic orbit imply the existence of other periodic orbits, and the structure that certain ones must have. Next we will look at an example and use a tool called kneading theory to explore how periodic points appear in a specific class of parameterized unimodal functions. The discussion will then shift to a more general setting of how the existence of certain periodic orbits in one system affects the existence of periodic orbits in nearby systems.

Our discussion will then turn to examining chaos and systems that make the transition to this particularly complex state. We will look at Robert L. Devaney’s definition of chaos [RD] and examine how the various parts of his definition relate to each other. Finally, we look at a particularly interesting subset of a class of functions known as transitional families as an illustration of the approach to chaos.
2 ~ Basic Ideas and Definitions

2.1 Overview

Since we are going to look at dynamical systems, we should first run through a brief discussion of dynamical systems in general, and what these things are. Generally, the principal problem in dynamical systems is the study of what happens to data as it is iterated through a function. This field is broken down into two large areas of study: continuous dynamical systems and discrete dynamical systems. The former is concerned with systems that arise from differential equations, while the later examines systems that come out of difference equations. A further classification can be made in the dimension of such systems. This paper looks at a special class of one-dimensional discrete dynamical systems, unimodal functions, with particular emphasis on a subset of a class of functions called transitional (or full) families. These systems are more conveniently analyzed because successive iterations of a one-dimensional system can be viewed in two dimensions and readily drawn; also a theorem due to Šarkovskii is very useful and only applicable in the one-dimensional setting.

Now, given a function, \( f(x) \), its graph is the usual way of analyzing its properties, but such a graph—as far as dynamical systems are concerned—is really the record of one iteration of the entire domain. In the study of dynamical systems we are more interested in what happens to a particular point as it is repeatedly iterated. After we know what happens to particular points, then we can begin to look at classifying different types of behavior and begin to understand what happens to the domain as a whole as it is repeatedly iterated.

2.2 Orbits

Our concern with these sets of iterated points brings us to define such a set.
**Definition 2.2.1:** Given a function $f(x)$ we can define an orbit of a point $x$ in the domain of $f$ as $O(x) = \{x, f(x), f^2(x), \ldots, f^n(x), \ldots\}$.

These orbits can be classified as periodic, eventually periodic, or nonperiodic.

**Definition 2.2.2:** An orbit is periodic if there exists an $n \in \mathbb{Z}^+$ such that $f^n(x) = x$; further if for all $k < n$ and $k \in \mathbb{Z}^+$, $f^k(x) \neq x$, then $x$ is said to be a periodic point of period $n$ or have a periodic orbit of period $n$.

**Definition 2.2.3:** An orbit is said to be eventually periodic of period $k$, if after some finite number of iterations, say $m$, $f^{k+m}(x) = f^{nk+m}(x)$ for all $x$.

All other points are nonperiodic, having nonperiodic orbits. An interesting and immediate result of having a nonperiodic orbit is that it can never overlap itself, thus each iteration generates a unique member of the orbit. This property gives us the fact that periodic and eventually periodic orbits have a finite number of unique members, while all other types of orbits have an infinite number of unique members. If $O(x)$ is an infinite set and the domain of $f$ is an interval of the real numbers, say $[0, 1]$, then members of this orbit must either have a convergent subsequence or be dense on some subinterval.

**Definition 2.2.4:** An orbit is said to be asymptotic if $O(x)$ has a convergent subsequence.

**Definition 2.2.5:** An orbit is wandering if it is none of the previous kinds listed in Definitions 2.2.2, 2.2.3, and 2.2.4; in other words it is not periodic or eventually periodic (hence has an infinite number of unique members) and has no convergence subsequence.

This kind of orbit must be dense on some subinterval, if the domain is a bounded interval of real numbers (the kind we will be looking at).
Before we leave this discussion on the types of orbits, we note that the Fixed Point Theorem is useful in establishing the existence of a fixed point or orbit of period 1. We will restate this theorem in a form that is useful for this paper.

The Fixed Point Theorem states that if a continuous function maps an interval onto itself, then there exists a point in that interval fixed by the function. Actually, the conditions of this theorem are overly broad; all one really needs is that \( f(J) \supseteq J \) rather than mapping onto \( J \), and one is still guaranteed a fixed point in \( J \) under \( f \).

2.3 Chaos

This field of study, being relatively new, has several differing definitions of chaos, each with their own following. The definition that this paper uses is Devaney's definition which captures both the complexity and structure of a chaotic system. Since chaos is not randomness but rather extremely complex structure, Devaney has three conditions in his definition: 'unpredictability, indecomposability, and an element of regularity' [RD]. His definition has three parts, each of which encompasses one of these elements. We will begin by defining the terminology used in each of the three parts. The definitions are all due to Devaney [RD].

By sensitive dependence on initial conditions we mean intuitively that a small error in the input will, under iteration, create unpredictable error, possibly quite large.

**Definition 2.3.1:** A system has *sensitive dependence on initial conditions* if there exists a \( \delta > 0 \) such that for all \( x \) in the domain of \( f \) and any neighborhood \( N \) of \( x \), there exists a \( y \) in \( N \) and an \( n \in \mathbb{Z}^+ \) such that \( |f^n(x) - f^n(y)| > \delta \). [RD]

By topological transitivity we mean any small piece of \( V \) will, under iteration, cover the entire space, \( V \).
**Definition 2.3.2:** A system is topologically transitive if given any two subsets of \( V \), say \( I \) and \( J \), there exists a finite number of iterations \( n \) such that \( f^n(I) \cap J \neq \emptyset \). [RD]

Of course, the density of periodic points simply means that given any open subset of \( V \) there exists within that subset a point whose orbit is periodic.

Now, we are ready for Devaney's definition of Chaos.

**Definition 2.3.3:** Let \( V \) be a set and \( f \) be a function mapping \( V \) into itself, then \( f \) is chaotic on \( V \) if

1) it has sensitive dependence on initial conditions,

2) it is topologically transitive, and

3) its set of periodic points is dense on \( V \). [RD]

We will examine this definition in more detail and discuss some interrelationships between the three parts in chapter five.

### 2.4 Transitional Families

Since we will be looking at a subset of transitional families for some of our extended examples, we will define just what we mean by a transitional family. Again, this definition uses a lot of terminology that we will define first. Here we will also define our primary object of study: the unimodal function.

**Definition 2.4.1:** A unimodal function is a function, \( f(x) \in C^1(I, I) \), such that \( f(0) = 0, f(1) = 0 \), and \( f \) has a unique critical point \( c \) with \( 0 < c < 1 \). [RD]

From kneading theory we get the kneading sequence, a very helpful construct in dynamical systems, which is also used in defining transitional families.
Definition 2.4.2: Divide $I$ into two intervals $I_0 = [0,c)$, and $I_1 = (c,1]$ and the function’s single critical point $\{c\}$, then a point’s itinerary is the sequence of 0’s, 1’s and c’s that corresponds to its position, in the sequence of iterations, being in either $I_0$, $I_1$, or $c$. [RD]

Definition 2.4.3: A function’s kneading sequence is the itinerary of the point, $c$, left-shifted by one element. [RD]

Definition 2.4.3 is due to Devaney and differs superficially from Milnor and Thurston’s original definition. [MT] In this paper derivatives at end points will be defined by the one-sided limit.

Definition 2.4.4: The Schwarzian Derivative of a function, $f$, is defined to be

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left( \frac{f''(x)}{f'(x)} \right)^2$$

for all $x$ in the domain of the function. The Schwarzian derivative at any critical point, $y$, is defined to be $\lim_{x \to y} Sf(x)$, and negative infinity is allowed as a value less than 0. [RD]

Now that we have all the pieces, we turn to defining the transitional family.

Definition 2.4.5: A transitional family is a family of functions that depend smoothly on a single parameter, $\lambda$, $\lambda_0 \leq \lambda \leq \lambda_1$. These functions are unimodal maps for $\lambda_0 < \lambda < \lambda_1$, with $f_{\lambda_0}(x) = 0$ for all $x$ in $I$. They have a negative Schwarzian derivative for all $\lambda_0 < \lambda < \lambda_1$ (ignoring any point that has an undefined derivative) and the kneading sequence $K(f_{\lambda_1}) = (100\bar{0}\ldots)$. [RD]

The Schwarzian condition may seem a bit strange and people like W. de Melo and S. van Strien [MS] have shown that for some of the primary theorems about the class of functions that satisfy this criterion, certain smoothness properties can be substituted. These authors prove that as long as our function is a member of $C^3$ and has non-flat critical points (meaning the function is $C^\omega$...
near each critical point and, additionally, the value of one of the derivatives at each critical point is nonzero—which basically is saying that close to each critical point the graph is not a horizontal line segment).[MS]

Unfortunately, the Schwarzian condition does more than that. It also links critical points and fixed points by lending its member functions the following property; the theorem and proof are due to Devaney [RD].

**Property 2.4.1:** If a unimodal function, \( f \), has a negative Schwarzian derivative, then the first derivative of \( f \) cannot have a critical point that is a negative relative maximum or a positive relative minimum.

**Proof:** Let \( Sf < 0 \) on \( I \). Let \( a \) be a point on the interior of \( I \) where \( f''(a) = 0 \) and \( f'''(a) \neq 0 \). Under these hypotheses, the Schwarzian derivative can be negative only if \( f''(a)/f'(a) < 0 \), which means that \( f'(a) \) and \( f'''(a) \) must have opposite signs. Hence \( f'(a) \) must be a positive relative maximum or a negative relative minimum. \( \square \)

This particular result of the Schwarzian condition will be used in some of following arguments. The hypotheses cannot be replaced by de Melo's smoothness condition spoken of previously. We could replace the Schwarzian condition with the smoothness properties and with the additional condition of the concavity changes being associated with critical points, but the Schwarzian condition makes a nice objective way of determining quickly if a function belongs to our discussion or not. So, since those who have come before have used this condition, we will use it, noting that there exist functions that satisfy all the necessary conditions but which are left out by the conventional definition of transitional families. We also note that Property 2.4.1 and de Melo’s smoothness condition does not imply a negative Schwarzian derivative.
2.5 Graphical Analysis

The concepts of orbits are sometimes easier to describe using graphical analysis, so we will stop for a moment and look at the graphs used to analyze the creation, position and paths of periodic orbits and their member points. A special kind of periodic point, the fixed point, is easy to locate on a graph; it is simply where the graph of the function, \( y = f(x) \), crosses the line \( y = x \). An interesting fact is that the graph of \( f^2(x) \) crosses the line \( y = x \) at all fixed points of \( f \) as well as all period two points, because \( f^2(x) \) maps all period two points of \( f \) onto themselves, making them fixed points of the function \( f^2(x) \). In general the fixed points of \( f^n(x) \) are all points of \( f \) that are either fixed, have period \( n \), or have a period of a factor of \( n \). For example \( f^6(x) \) has as its fixed points those which under \( f \) are fixed, period two, three, or six.

For this paper the pictorial examples that we will be using (unless otherwise noted) will be from the quadratic family. This family of functions is \( f_\lambda(x) = \lambda x(1 - x) \), where \( \lambda_0 = 0 \leq \lambda \leq 4 = \lambda_1 \). As we will show with the following property, this is a transitional family.

**Property 2.5.1:** The quadratic family is a transitional family for \( 0 \leq \lambda \leq 4 \).

**Proof:** Clearly this function is unimodal, and \( f_{\lambda_0}(x) = 0 \). Now, \( f_{\lambda_1}(x) = 4x(1 - x) \) making \( f'(c) = 1 \) (\( c \) being the function's unique critical point), and \( f(1) = 0 \), so \( K(f_{\lambda_1}) = (10\overline{0}) \ldots \). Since the third derivative of a quadratic polynomial is zero, the first term in the Schwarzian derivative drops out leaving the second term, which is the negative of a square, thus always negative. So \( Sf < 0 \), for \( \lambda_0 = 0 < \lambda \leq 4 = \lambda_1 \).

Another tool of graphical analysis is the web graphs. This method uses the fact that if one draws a horizontal line from one point on the graph to the line \( y = x \) then a vertical line back to the graph, one has actually once iterated the output of the function. See Figure 1 for an example of a
web graph of the critical point having period four. This graph not only makes it easy to see the periodic behavior but also the location of the members of this particular orbit.

2.6 Types of Bifurcations

Next we will look at just how these periodic points appear. As previously mentioned, we will be looking at the transition of unimodal functions from a simple state to a chaotic one. By Devaney's definition the chaotic state requires that periodic points be dense on the function's domain. How are these periodic orbits created? They arise from an occurrence called a bifurcation.

A bifurcation occurs in a parameterized function, like a transitional family, when the parameter, \( \lambda \), changes the function enough to form new periodic points. These bifurcations take on only two forms in the one-dimensional setting: the saddle-node type and the period-doubling type. These bifurcations are analyzed by looking at the formations of fixed points in \( f_\lambda^n(x) \) with the appropriate value for \( n \).

Saddle-node bifurcations occur when a loop of the graph cross the \( y = x \) line as \( \lambda \) changes. When this occurs, the function has a slope of one. The reason for this fact is that for a piece of the graph to cross the line it must first touch at only one point, at which time it must be tangent to the line, hence have a slope of one. Figure 2 shows the development of a saddle-node bifurcation in the \( f_\lambda^3(x) \)
member of the quadratic family as $\lambda$ progresses from 3.82 to 4.84. The hallmark of this type of bifurcation is that a new fixed point suddenly appears then splits into two, forming twin orbits of identical period—we will refer to these twin orbits as an orbit-pair. Note, we are looking at the formation of fixed points in $f^3_\lambda(x)$, which are (in this case) period three points in $f_\lambda(x)$.

The second type of bifurcation differs significantly from the first. The period-doubling bifurcation occurs around an existing periodic point. The two new points have period double that of the birthing point and these new points actually belong to the same orbit. This type of bifurcation occurs where the graph already crosses the line $y = x$, but kinks and creates two new bends in the curve. One similarity that the period-doubling bifurcation has with the saddle-node bifurcation is that at the moment of the bifurcation the graph of the function is tangent to the line $y = x$; thus having a slope of one. Figure 3 shows a period-doubling bifurcation occurring in $f^2_\lambda(x)$ as $\lambda$ runs...
from 2.9 to 3.1. This bifurcation is occurring around a fixed point (period of one), giving the new
points a period of two. Again, note that we are looking at the formation of fixed points in \( f_{\lambda}^2(x) \),
which are, in this case, a fixed point and a pair of period two points in \( f_{\lambda}(x) \). Further note that the
birthing point is not destroyed.

\section*{2.7 Kneading Theory}

Now, we will look at Kneading Theory, a very useful tool in analyzing one-dimensional
dynamical systems, which we will use in greater detail in section 3.3. We looked at this idea briefly
when we defined the kneading sequence, now we will look at it more carefully before continuing our
discussion. The following definition is due to Devaney [RD]. Since we are looking only at
unimodal functions, we will present this theory as it applies to systems with only one critical point,
although it can be generalized to systems with any number of critical points. As mentioned before,
this tool has at its heart a notation that traces the path of any point through its orbit. Instead of
actually listing every value the point takes on in its orbit, all that is recorded is on which side of the
critical point, \( c \), the points lie. Recall, that after we defined two sub-intervals of \( I: \)
\( I_0 = [0, c) \), and \( I_1 = (c, 1] \), we said that the itinerary of a point \( x, S(x) \), is the sequence of 0's, 1's and
c's that corresponds to its iteration value being in either \( I_0 \), \( I_1 \), or \( \{ c \} \), respectively. For example,
referring back to Figure 1, the itinerary of \( c \), shown in the web graph, would be
\( S(c) = (c10\overline{1}c10\overline{1}\ldots) \) (the over-bar indicating an endlessly repeating sequence). Note, as mentioned
before, the kneading sequence of this example would be the above itinerary left-shifted one, i.e.,
\( K(f) = S(f(c)) = (101c\overline{01}c\ldots) \). Useful properties of these sequences are readily apparent:
periodic orbits have periodic sequences like the one shown (though the converse is not true), the
length of the period is easily determined, and orbits that pass through the critical point are easy to identify.

In addition to the sequence, Devaney defines an ordering of these itineraries.

**Definition 2.7.1**: Let \( s = (s_1 s_2 s_3 \ldots s_n s_{n+1} \ldots) \) and \( t = (t_1 t_2 t_3 \ldots t_n t_{n+1} \ldots) \). We define the ordering, '\(<\)': at the first entry where there is a discrepancy in the two itineraries (say the \( n \)-th entry \( s_n \neq t_n \)), if the number of earlier 1's is even and \( s_n < t_n \) or the number of earlier 1's is odd and \( s_n > t_n \), then \( s < t \) [RD].

Though this definition seems awkward, it is defined this way because this ordering mirrors the ordering of real numbers; that is if \( a < b \), then \( a \)'s itinerary will be less than or equal to \( b \)'s itinerary. We prove this fact in the following theorem and proof due to Devaney.[RD]

**Theorem 2.7.1**: Let \( x \) and \( y \) be in \( I \).

1) If \( S(x) < S(y) \), then \( x < y \).

2) If \( x < y \), then \( S(x) \leq S(y) \).

Note that the symbol \( \leq \) means \( < \) or \( = \).

**Proof**: We will use an induction argument to prove part one. Let \( S(x) = (s_1 s_2 s_3 \ldots) \) and \( S(y) = (t_1 t_2 t_3 \ldots) \). Assume that \( S(x) < S(y) \). Our induction will be on the place of the first discrepancy. We will start with the case of \( n = 1 \), i.e., when the discrepancy occurs in the first entry. In this case \( x \) and \( y \) are either in different disjoint intervals \([0,c)\) and \((c,1]\), or one of them is \( c \) and the other is not. Since there are no (read an even number of) 1's before the discrepancy, the order matches the natural order of the first element. Clearly this means that \( x < y \).

Now we assume our theorem true for the cases where the first discrepancy occurs within the first \( n - 1 \) entries, and will prove it true for when the first discrepancy occurs in the \( n \)-th place. First,
we assume that \( n \) is not one, so the first element is not a discrepancy thus \( s_1 = t_1 \). If we apply a left-shift to both sequences, we will not lose a discrepancy and the actual discrepancy is moved to the \( n - 1 \) place. So we have \( S(f(x)) = (s_2, s_3, s_4, \ldots) \) and \( S(f(y)) = (t_2, t_3, t_4, \ldots) \). Next we look at these new sequences, and see how they compare under our ordering. We have three cases depending on the value of the first elements being 0, \( c \), or 1. Actually, if the first element was \( c \), then we actually have \( x = y \) since both \( x \) and \( y \) must be \( c \). Now, if the common first element was a zero, we have not changed the number of one’s lying before the discrepancy, thus since \( S(x) < S(y) \) we have \( S(f(x)) < S(f(y)) \) with a discrepancy in the \( n - 1 \) place. By our induction hypothesis, \( f(x) < f(y) \), but we know more. Since the first entry of both itineraries was zero, we know that \( x \) and \( y \) are both in the interval \( [0, c) \). Also, since \( f \) is unimodal and thus increasing on this entire interval, \( x < y \).

Suppose, on the other hand, that the common first entry was a one. Then we have changed the number of one’s before the first discrepancy, making \( S(f(x)) > S(f(y)) \). Now, by our induction hypothesis, we have \( f(x) > f(y) \). A similar analysis reveals that both \( x \) and \( y \) are now in the interval \( (c, 1] \), where \( f \) is decreasing; hence, \( x < y \). Thus, part one of our theorem is proved.

Part two of this theorem is immediate. Let \( x < y \) and suppose, contrary to our theorem, that for some choice of \( x \) and \( y \) we have \( S(x) > S(y) \). Then by part one \( S(x) > S(y) \) implies that \( x > y \), which is a contradiction. Thus, \( S(x) \not> S(y) \) which means that \( S(x) \leq S(y) \) for any \( x \) and \( y \). and we are done.

An immediate and important result of the preceding theorem lies at the heart of kneading theory. In any unimodal function, the kneading sequence must be the sequence of highest order of any sequence generated by \( f(x) \).

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Property 2.7.1: For all $x$ in $[0,1]$, $S(f(x)) \leq K(f)$.[RD]

This property follows from our theorem and the fact that in a unimodal map $f(x) \leq f(c)$ for all values of $x$. 
3 ~ The Formation and Structure of Periodic Points

3.1 Šarkovskii’s Theorem

All of our discussions in this section will have as one of their foundations, perhaps the most important theorem of this topic: Šarkovskii’s Theorem. This theorem states that the existence of periodic orbits of one length implies the existence of orbits of other lengths.

**Definition 3.1.1**: The Šarkovskii ordering is the following ordering of all positive integers:

\[ 3 > 5 > 7 > \ldots > 2 \cdot 3 > 2 \cdot 5 > \ldots > 2^2 \cdot 3 > 2^2 \cdot 5 > \ldots > 2^3 > 2^2 > 2 > 1. \]

The highest number in this ordering is 3, and 1 is the lowest. The ordering descends from three through all odd integers, then through all odd integers multiplied by 2, then all odd integers multiplied by \(2^2\) and so on until all that is left are the integers that are powers of 2 only, these are listed in descending order to 1.

**Theorem 3.1.1 (Šarkovskii’s Theorem)**: If \( f : \mathbb{R} \rightarrow \mathbb{R} \) is continuous, then the existence of a period higher in the Šarkovskii ordering implies the existence of all periods lower in the ordering.

For example, fixed points (period 1) imply nothing by this theorem, but the existence of period 2 implies the existence of period 1. Curiously, the existence of period 3 implies all possible periods exist and the existence of any odd period implies the existence of an infinite number of different kinds of periods, namely all even periods.

The following proof of Šarkovskii’s theorem is used by de Mello [WM], who attributed it to Block. It uses a construction called a Markov graph, so we digress for a moment to look at these graphs.
**Definition 3.1.2:** Consider a function $f$ mapping $I$ into itself, and let $\{I_i\}_{i=1}^k$ be a partition of $I$ by subintervals. A *Markov graph* is a set of vertices labels by the $I_i$ and a set of directed edges with the following property: $I_n \rightarrow I_m$ if $f(I_n) \supset I_m$.

It is possible for an edge to map a vertex back onto itself; in that case we note by the Fixed Point Theorem that there must be a fixed point in that interval since $f(I_n) \supset I_n$ implies that there must be a fixed point of $f$ in $I_n$. A powerful property of these graphs is that if one can trace a path through the graph that returns to a starting vertex, in $j$ steps, then $f^j(I_n) \supset I_n$, hence there exists a fixed point of $f^j$ in $I_n$.

Here is an illustrative example from De Melo.[WM] Let $f$ be a continuous function mapping the unit interval into itself with $f(p_1) = p_2$, $f(p_2) = p_3$, $f(p_3) = p_1$ such that $0 < p_1 < p_2 < p_3 < 1$. This function has a periodic orbit of period three (hence, as we will soon show, orbits of all periods). If we were to label the following intervals: $I_1 = [p_2, p_3]$ and $I_2 = [p_1, p_2]$. Now, by appealing to continuity of our function, we have $f(I_2) = I_1$, so $f(I_2) \supset I_1$. We also have $f(I_1) = I_1 \cup I_2$, so $f(I_1) \supset I_1$ and $f(I_1) \supset I_2$. With this information we can construct the following Markov graph:

![Markov Graph Diagram]

De Melo’s proof of Šarkovskii’s Theorem takes several steps and five intermediate lemmas.

**Proof:** Let $x$ be a periodic point of period $n$ of a continuous function, $f$, that maps the interval $I$ into itself. Let $O(x) = \{x_0, x_1, \ldots, x_{n-1}\}$, the orbit of $x$, such that $x_i < x_{i+1}$. Define $J = [x_0, x_n - 1]$ and partition it by the series of closed intervals: $[x_i, x_{i+1}]$, which are the vertices of a Markov graph. The five lemmas are used to prove that this graph has certain kinds of vertices.
Lemma 3.1.1a: The above Markov graph has a vertex, call it \( I_1 = [x_a, x_{a+1}] \), such that \( f(I_1) \supset I_1 \). Also, \( f(x_{a+1}) \leq x_a < x_{a+1} \leq f(x_a) \).

In other words, the graph has a vertex (interval) with an edge that connects this interval with itself, and \( f \) maps this interval over itself in reverse orientation.

**Proof:** Note that the entire orbit of \( x \) is contained within \( J \). So, \( f(x_0) > x_0 \) because the only other possibility is that \( f(x_0) = x_0 \) which contradicts the fact that \( x \) has period \( n \). Similarly, \( f(x_{n-1}) < x_{n-1} \). Since at least one of the members of the orbit maps to something larger, take the largest of these members: \( x_a = \max \{ x_i \mid f(x_i) > x_i \} \), \( 0 < a < n - 1 \). Note that \( x_a \) cannot be \( x_{n-1} \) since this member of the orbit does not have the required property. Such a selection for \( a \) gives us the necessary properties for \( I_1 = [x_a, x_{a+1}] \). We have \( f(x_a) \geq x_{a+1} \). By our construction \( f(x_a) > x_a \), but \( f(x_a) \) is a member of \( x \)'s orbit hence it must be one of the \( x_i \) that are bigger than \( x_a \), the smallest of these being \( x_{a+1} \).

Finally, \( f(x_{a+1}) \geq x_a \) because \( x_a \) was the largest member of the orbit with the desired property. So, \( x_{a+1} \) must map to a member of the orbit smaller than itself; \( x_a \) being the largest candidate. Note that \( f(I_1) \supset I_1 \).

Throughout the rest of the preliminary lemmas and the actual proof of Šarkovskii's Theorem, the interval \( I_1 \) will mean the interval as we constructed it in Lemma 3.1.1a. The next lemma will show that there is a path through the Markov graph from \( I_1 \) to any other vertex.

**Lemma 3.1.1b:** Let \( I_1 \) be the vertex of the Markov graph defined above in Lemma 3.1.1a. For any vertex, \( K \), of the graph there exists a path connecting \( I_1 \) to \( K \).

**Proof:** Define \( V_i \) to be the set of all vertices (intervals) such that there exists a path from \( I_1 \) along \( i \) edges. Since \( I_1 \) has an edge back to itself, any path with \( i \) edges to a particular vertex can
be extended to a path with \( i + 1 \) edges to the same vertex, i.e. the path \( I_1 \rightarrow K_2 \rightarrow \ldots \rightarrow K_i \) can be naturally extended to \( I_1 \rightarrow I_1 \rightarrow K_2 \rightarrow \ldots \rightarrow K_i \). So, \( V_i \subset V_{i+1} \).

Further we will define \( U_i \) to be the set consisting of all points in the union of all members (intervals) of \( V_i \). Since \( V_i \subset V_{i+1} \), we have \( U_i \subset U_{i+1} \). Now, suppose that there is a member of \( V_i \), say \( \hat{K} \), such that \( f(\partial \hat{K}) \) contains points that do not belong to \( U_i \). Note that \( \partial \hat{K} \) means the boundary of \( \hat{K} \). We can conclude that \( f(\hat{K}) \) contains a vertex (interval) not in \( V_i \) and, naturally, \( V_i \neq V_{i+1} \). Now if our supposition was true for all \( i \), we can construct an infinite, strictly nested sequence of \( V_i \)'s.

But, we have only a finite number of vertices (intervals), hence only a finite number of the \( V_i \)'s can be distinct. So, there must be a \( j \leq n - 1 \) such that our supposition fails and \( V_j = V_{j+1} \). Thus, under repeated iterations of \( f \), \( V_j \) does not change, so \( U_j \) does not change either. But the only set that contains the orbit of \( x \) and is invariant under \( f \) must be the entire interval, \([x_0, x_{n-1}]\). So, \( U_i = [x_0, x_{n-1}] \) and \( V_j \) contains all the vertices (intervals) of the system. Hence all vertices in the graph must be connected by a path to \( I_1 \).

Now the proof turns to looking for existence of specific periods under specific conditions. The next lemma proves that if we suppose that \( I_1 \) is the only vertex in the graph, then \( x \) has a point of period 2. The following two lemmas address what happens if \( x \) has an odd period or an even period.

**Lemma 3.1.1c**: Suppose that \( I_1 \) is connected only to itself in the Markov graph. Then \( f \) maps all points to the left of \( I_1 \)'s interior to those points to the right of \( I_1 \)'s interior, and vice-versa. Also, the period of \( x \) is even and \( f \) has a periodic point of period 2.
Proof: Recall that our construction, \( I_1 = [x_a, x_{a+1}] \), in Lemma 3.1.1a was made such that \( f(x_a) \geq x_{a+1} \). Now, suppose that there is (contrary to what we wish) a point \( x_i < x_a \) such that \( f(x_i) \leq x_a \), i.e., a point on the left of the interval that stays there under one iteration. We will find the largest such point and derive a contradiction.

Let \( x_b = \max \{ x_i \mid x_i < x_a \text{ and } f(x_i) \leq x_a \} \). Since \( x_b \) is the largest member of the orbit of \( x \) for which \( f(x_b) \leq x_a \), then \( f(x_{b+1}) > x_a \) must be true. Since there is no member of the orbit between \( x_a \) and \( x_{a+1} \), we have \( f(x_{b+1}) > x_a + 1 \). But these facts give us \( f([x_b, x_{b+1}]) \supseteq I_1 \). This statement contradicts our hypothesis that no edge connects \( I_1 \) to anything but itself since it implies that there exists an edge connecting \([x_b, x_{b+1}]\) to \( I_1 \), while \([x_b, x_{b+1}] \neq I_1 \).

By an identical argument, only this time we will make the construction \( x_c = \min \{ x_i \mid x_i > x_a + 1 \text{ and } f(x_i) \geq x_a + 1 \} \), we arrive at the similar conclusion that \( f(x_{c+1}) \leq x_a \) while \( f(x_c) > x_{a+1} \), making \( f([x_{c+1}, x_c]) \supseteq I_1 \), and we get another contradiction.

So, all points to the left of the interior of \( I_1 \) map to the right of the interior and vice versa. This fact forces the period of \( x \) to be even. We know that \( x \) is periodic by hypothesis. Since the exterior and boundary of \( I_1 \) contains all the members of \( x \)'s orbit, we can see that under each iteration the orbit jumps from one side of our interval to the other. To count the period of the orbit we must end up on the same side that we started, which is only possible under an even number of iterations.

Lastly, we will show that there must exist an point, not necessarily \( x \), that has period of length 2. Let \( J_0 = [x_0, x_a] \) and \( J_1 = [x_{a+1}, x_{a+1}] \). By the first part of this lemma, we know that \( f(J_0) \supseteq J_1 \) and \( f(J_1) \supseteq J_0 \) thus \( f^2(J_0) \supseteq J_0 \), giving us a point of period 2. We do not have a fixed point since the interval is mapped into \( J_1 \) before returning. \( \square \)
Lemma 3.1.1d turns to looking at the structure of the Markov graph for a function with odd periods.

**Lemma 3.1.1d:** Assume that \( f \) has periodic points of odd period. Let \( n > 1 \) be the smallest such period, and let \( x \) be a point of period \( n \). Then the following is true:

1. There exists an index scheme such that every vertex has an edge mapping to the vertex having the next larger index with the vertex having the largest index mapping to the vertex with the smallest index and the smallest to itself:
   \[
   I_1 \to I_2 \to \cdots \to I_{n-1} \to I_1 \to I_1.
   \]
2. No edge maps \( I_j \to I_{j+k} \) for all \( k > 1 \).
3. \( I_{n-1} \) has an edge mapping to every odd-numbered vertex.

**Proof:** We will start by proving (1). Note, of course, that \( I_1 \) is the interval given by Lemma 3.1.1a so we already know that it has an edge mapping to itself. By Lemma 3.1.1b we know that there exists a path of edges from \( I_1 \) to all vertices. Notice that since \( x \) has period of \( n \), a loop of length \( n \) must exist from \( I_1 \) and back to \( I_1 \). Let \( k > 1 \) be the smallest integer such that we get a path back to \( I_1 \) i.e. \( I_1 \to \cdots \to I_k \to I_1 \). We claim that \( k \) must be \( n - 1 \). If \( k < n - 1 \) that would create a loop smaller than \( n \). If \( k \) is odd, we have caused the existence of a point of odd periodic of length less than \( n \) contradicting the minimality of \( n \). If \( k \) is even, the loop \( I_1 \to \cdots \to I_{n-1} \to I_1 \to I_1 \) has odd length still less than \( n \) and still makes a contradiction. Thus we have proved that for some labeling of the vertices we get part (1).

Part (2) follows immediately by noting that since \( n \) is the smallest odd period, if any vertex (other than \( I_{n-1} \)) maps to two vertices it effectively cuts the path length to something smaller and creates a point of odd period less than \( n \). Say you cut a single vertex out of the chain, then you have
$I_1 - I_3 - \cdots - I_{n-1} - I_1$, a path of odd length $n - 2$. This construction contradicts the minimality of $n$.

Next we look at (3). What we will do here is notice how the $I_i$'s are nested and carefully label them in such an order that we get what we want: $I_{n-1}$ having an edge mapping to every odd numbered vertex.

Note that by our construction of $I_1$, we have $f(x_a) \geq x_{a+1}$ and $f(x_{a+1}) \leq x_a$. Now, $x$ is of odd period so it cannot have period two and thus $f(x_a) \neq x_{a+1}$ or $f(x_{a+1}) \neq x_a$ for if we have equality in both places we have $x$ with a period of 2. Suppose the latter is true (if the former is true, the necessary construction is essentially the mirror image of the following), then $f(x_{a+1}) < x_a$. This fact forces $f(x_a) = x_{a+1}$ and $f(x_{a+1}) = x_{a-1}$.

If we label $[x_{a-1}, x_a]$ as $I_2$, we note that $f(I_1) \supset I_1 \cup I_2$, so we have an edge connecting $I_1$ to itself and another edge connecting it to $I_2$. If $f(x_a)$ or $f(x_{a+1})$ were to equal anything else $f$ would map the first interval, $I_1$, over three intervals and contradict Part (2) by allowing edges to connect $I_1$ to two other vertices other than itself.

Essentially we repeat the above argument until we label all of the intervals. We know that $f(x_a) = x_{a+1}$, so $f(x_{a-1}) = x_{a-2}$, because otherwise $f$ will map $I_2$ over more than one interval, and this situation would cause more than one edge to map out from $I_2$, contradicting Part (2). (Unless, of course, $n$ was 3, and then $f(x_{a-1}) = x_a$). So, we label $[x_{a+1}, x_{a+2}]$ as $I_3$.

By repeating this argument one more time we find that $I_3$ maps to the next interval to the left of $I_2$, we call that one $I_4$ and so on, ordering the intervals thus $I_{n-1}, \ldots, I_2, I_1, I_3, \ldots, I_{n-2}$. Note that since $n - 1$ is even it will end the series on the left.
By this construction, all even-numbered intervals, $I_{2i}$, map to the next of the indexed intervals, $I_{2i+1}$, in such a way that the lower endpoint of $I_{2i}$ maps to the upper endpoint of $I_{2i+1}$. Hence for the interval $I_{n-3}$, we note that $f(x_1) = x_{n-1}$, but $x_1$ is also the upper endpoint of $I_{n-1}$, so this interval gets mapped over $x_{n-1}$ as well. We also note that $I_{n-1}$ maps over $I_1$ as well, by Part (1). So, by the continuity of $f$, $f(I_{n-1}) = [x_n, x_{n-1}]$. In other words $f$ maps $I_{n-1}$ over all the odd-numbered intervals (vertices). So there exists an edge from $I_{n-1}$ to all odd-numbered vertices and we have proved (3) and finished the lemma.

This lemma has a corollary that finally begins to address the theorem at hand.

**Corollary 3.1.1a:** If $f$ has a point of odd period $n$, then $f$ has periodic points of all periods greater than $n$ and all even periods less than $n$.

**Proof:** To prove this corollary we consult the Markov graph constructed in the previous lemma. If $m$ is larger than $n$, we simply look at the path where we add enough loops through $I_1$ to lengthen our path from $n$ to $m$: $I_1 \to \cdots \to I_1 \to \cdots \to I_{n-1} \to I_1$. If $m = 2i < n$, then we start at $I_{n-1}$, run down the edge to the appropriate odd vertex and loop back to $I_{n-1}$: $I_{n-1} \to I_{n-2i} \to I_{n-2i+1} \to \cdots \to I_{n-1}$.

We look at one last lemma that gives us information about the situation when an even period exists.

**Lemma 3.1.1e:** If $f$ has a periodic point of even period, then $f$ must have a point of period 2.

**Proof:** We begin by looking at the period of the smallest periodicity in $f$. Let $n$ be the smallest period in $f$. Necessarily $n \geq 2$, assume it is strictly greater than two. We know that $n$ must be even because if it were odd, $f$ would have a point of period 2 by our previous corollary and
contradict our assumption. Using the contrapositive of Lemma 3.1.1c, we note that there must exist a vertex $I_k$ such that $f$ maps this vertex to $I_1$. If no such $I_k$ existed, then by Lemma 3.1.1c, $f$ would have a point of period 2 contradicting our assumption.

We let our $I_k$ have the smallest index of the vertices that map to $I_1$. So, the Markov graph has the path $I_1 \rightarrow I_2 \rightarrow \cdots \rightarrow I_k \rightarrow I_1$. Duplicating the argument in Lemma 3.1.1d, where we appealed to the minimality of $n$, we can conclude that $k$ must be $n - 1$. We also conclude using the same argument that Part (2) of Lemma 3.1.1d holds as well, namely no edge maps $I_j \rightarrow I_{j+1}$ for all $i > 1$. Since $n$ is even, when we duplicate our argument for Part (3) of Lemma 3.1.1d our last interval, $I_{n-1}$ sits at the extreme right of our series of intervals: $I_{n-2}, \ldots, I_2, I_1, I_3, \ldots, I_{n-1}$.

Looking back at that construction, we note now that every upper endpoint of an odd interval maps to the lower endpoint of the next indexed (even) interval. So, looking at $I_{n-3}$, we note that for its upper endpoint, $x_{n-2}, f(x_{n-2}) = x_0$, the lower end point of $I_{n-2}$. Again since $x_{n-2}$ is also the lower endpoint of $I_{n-1}$, we have $f$ mapping this interval over $x_0$. Finally, noting that $f$ maps $I_{n-1}$ over $I_1$, we conclude that $f(I_{n-1}) = [x_0, x_{n-1}]$. In other words $f$ maps $I_{n-1}$ over all even indexed intervals (vertices), hence there must be edges connecting this vertex with all even vertices. But this leads to the path $I_{n-1} \rightarrow I_{n-2} \rightarrow I_{n-1}$, which gives a point of period 2. \qed

We are now ready to prove Šarkovskii’s Theorem. The proof of Šarkovskii’s Theorem has two cases: where the given period has a length equal to a power of 2 and where the given period has all other lengths.

Case 1: $f$ has a point of period $n = 2^k$. Let $m$ be any number less than $n$ under the Šarkovskii ordering. So, $m = 2^l$, with $l < k$. We have two sub-cases, if $l$ is zero, ($m$ being one) and if $l$ is not
immediately from Lemma 3.1.1a since there exists a sub-interval, $I_1$, such that $f(I_1) \supset I_1$, which gives us a fixed point.

If $I \neq 0$, then we make a clever construction and apply Lemma 3.1.1e. Let $g = f^{m/2}$. Now $g$ has a periodic point of period $2^{k/(m/2)}$, which is $2^{k/(2^l/2)}$ or $2^{k-l+1}$. By Lemma 3.1.1e, $g$ has a point of period 2, which means that $f$ has a point of period $2(m/2)$, that is a point of period $m$.

Case 2: $f$ has a point of period not a power of 2; a period of $n = p2^k$, where $p$ is a positive integer greater than one and $k$ is a non-negative integer. Next we look at an $m$ less than $n$ on the ordering and we arrive at three sub-cases depending on the form of $m$. We have only two possibilities for $m$: it could be a larger integer or a power of 2 less than $n$. The first possibility gives rise to our first two sub-cases: $m = q2^k$, where either (a) $q$ is odd and $q > p$, or (b) $q$ is even. The third sub-case, (c), arises when $m = 2^l, l \leq k$. If one were to look at the Šarkovskii ordering and notice that it has an infinite series of sub-sequences of ascending integers followed by a descending sequence of the powers of 2, case (a) takes care of all numbers lower than $n$ in $n$’s sub-sequence, case (b) takes care of all of the lower ascending sub-sequences and part of the sequence of powers of 2, while (c) accounts for the rest of the powers of 2.

First, we look at sub-cases (a) and (b). Again using a clever construction and appealing to Lemma 3.1.1d will give us our answer. This time we consider $g = f^{2^k}$. Now, $g$ has a periodic point of period $p2^{k/2^k}$ or $p$. Since $p$ is odd, we appeal to the corollary of Lemma 3.1.1d and note that $g$ must have points with period greater than $p$ and all even periods less than $p$. Thus, $g$ has a point with period $q$ no matter which case, (a) or (b), we consider, since $q$ is either greater than $p$ or is even. Since $g$ has a point of period $q$, $f$ has a point of period $q(2^k)$ or $m$. 

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Finally, we look at sub-case (c). We continue to examine our construction of $g$ and notice that (b) gave us the existence of all periods of a power of 2 greater than $2^k$. So, we know that $f$ has a point of period $2^{k+1}$ (case (b) with $q = 2$). Case 1 now tells us that $f$ has points of periods of powers of 2 less than $2^{k+1}$. Since $l \leq k$, we know that $f$ has a point of period $2^l$.

Thus Šarkovskii's theorem is proved.

3.2 Štefan’s Theorem

Next we will look at a theorem by Štefan that proves that if a system has an odd period as its left-most period in the Šarkovskii ordering, that orbit has a specific structure, namely:

$$f^{-1}(x) < \ldots < f^2(x) < f(x) < f^3(x) < \ldots < f^n(x) = x$$

or

$$x = f^n(x) < \ldots < f^3(x) < f(x) < f^2(x) < \ldots < f^{-1}(x).$$

This result is interesting by itself, but it will also be used in Lemma 4.2.1c.

We will start by defining some of the symbols used by Štefan, then move on to some preliminary lemmas and then to the main theorem; all of these results and their proofs are due to Štefan [PS].

First, note that unless a point is fixed under $f$ it must map to some value larger or smaller than itself, we will partition these non-fixed points into two sets. We will define these sets and a few other objects used in the following theorem and associated lemmas.
Definition 3.2.1: Let \( \bar{U} = \{x \mid f(x) > x\} \) and \( \bar{D} = \{x \mid f(x) < x\} \). Let \( \bar{U}_x = O(x) \cap \bar{U} \) and \( \bar{D}_x = O(x) \cap \bar{D} \). For a given periodic orbit, \( O(x) \), let \( \omega^U = \max \bar{U}_x \), \( \omega^D = \min \bar{D}_x \), \( \omega_{\min} = \min O(x) \), and \( \omega_{\max} = \max O(x) \).

The following theorem is essentially concerned with resolving the following question: Is an orbit member's image, under \( f \), larger or smaller than itself? With this idea in mind, we make the following observation.

Property 3.2.1: Suppose that there does not exist a fixed point of \( f \) between any of the following points: \( x, f(x), \ldots, f^n(x) \). If \( x < f(x) \), then \( f^i(x) < f^j(x) \) for all \( 0 \leq i < j \leq n \). Similarly, \( x > f(x) \) implies that \( f^i(x) > f^j(x) \) for all \( 0 \leq i < j \leq n \).

Proof: The proof of this property follows immediately from the observation that since there are no fixed points between \( x, f(x), \ldots, f^n(x) \), all these members of the orbit must be on the same side of the line \( y = x \). If they are above the line, that means that each point maps to something larger than itself. If they lie below the line, then each point maps to something smaller than itself. \( \square \)

Now, we will begin to assemble the preliminary lemmas. Before we do, we comment that the entire proof has two cases which are proven identically by a symmetrical argument.

**Lemma 3.2.1a:** If there exist points \( a, b, \) and \( c \) such that \( f(b) < c < a < b \leq f(a) \) (or \( f(b) > c > a > b \geq f(a) \)) and \( f(c) = c \), then there exists points of all possible periods under \( f \).

**Proof:** Since \( f(a) \geq b \) and \( f(c) = c \), we have \( f([c,a]) \supset [a,b] \). Thus there is a point \( d \in (c,a] \) such that \( f(d) = b \). If we label \( I_1 = (c,d) \) and \( I_2 = (d,b) \), then \( f(I_1) \supset I_1 \cup I_2 \) and \( f(I_2) \supset I_1 \cup I_2 \). From these facts we can construct the following Markov graph: \( I_1 \rightarrow I_1 \rightarrow I_2 \rightarrow I_1 \), thus there exists \( J \subset I_1 \) such that \( f^2(J) \subset I_2 \) and \( f^3(J) \supset I_1 \supset J \). This fact implies the existence.
of a point in $J$ with period 3. We note that it cannot be fixed since it leaves the interval $I_1$ during its orbit. By Šarkovskii's Theorem there exists points of all periods.

Lemma 3.2.1b: Let $O(x)$ be a periodic orbit of $f$. If $f$ has a fixed point $c$ such that $\omega_{\min} < c < \omega^U$ (or $\omega^D < c < \omega_{\max}$), then $f$ has points of all possible periods.

Proof: Let $\omega_{\min} < c < \omega^U$ and a point, $c$, fixed under $f$. Let $A = \{ y | y \in \tilde{U}_x, y > c \}$. Pick a point, $a$, such that $f(a) = \max \{ f(y) | y \in A \}$. Let $d \in [c, a)$ be the largest fixed point on that interval. Thus $c < d < a$. Since $\omega_{\min} < c$ there exists a positive integer, $p$, such that $f^p(a) = \omega_{\min}$. So, $\{ x, f(x), f^2(x), \ldots \} \subseteq [d, f(a)]$, and there exists a point, $b \in O(x) \cap [d, f(x)]$ such that $f(b) \in [d, f(a)]$. Note that if $b \in A$, then $f(b) < f(a)$, since $a$ was chosen to be the member of the orbit within $A$ that maps to the highest value. Note also that if $b \in \tilde{D}_x$, then $f(b) < b \leq f(a)$.

Since $f(b) < f(a)$ and $f(b) \in [d, f(a)]$, we must have $f(b) < d$.

If $b < a$, we use the fact that $f(b) < d < b$ and $a < f(a)$ to get $f([b, a]) \supseteq [b, a]$. But this fact implies that there is a fixed point of $f$ in $[b, a]$, which contradicts the fact that $d$ was the maximum fixed point in $[c, a]$. Thus, we must have that $b > a$, giving us $f(b) < d < a < b \leq f(a)$ with $d$ fixed under $f$. So, by Lemma 3.2.1a, we have that $f$ has points of all possible periods.

Lemma 3.2.1c: Let $O(x)$ be a periodic orbit of $f$. If $\omega^D < \omega^U$, then $f$ has points of all possible periods.

Proof: By our definition, $f(\omega^D) < \omega^D$ and $f(\omega^U) > \omega^U$. Thus, $f((\omega^D, \omega^U)) \supseteq (\omega^D, \omega^U)$, and there is a fixed point of $f$ in the interval $(\omega^D, \omega^U)$. So we have a fixed point between $\omega_{\min}$ and $\omega^U$, hence by Lemma 3.2.1b we have that $f$ has points of all possible periods.

Now we are ready for the main preliminary lemma that will be used several times within the body of the theorem's proof.
Lemma 3.2.1d: Let $O(x)$ be a periodic orbit of $f$ with odd period $n, \ n \neq 1$. Suppose there is some smallest positive integer, $q$, for which $f^q(b) \leq a < f(a) \leq b < f(b)$ for some $a$ and $b$ in $O(x)$, then for all $k \geq q$, there exists a point, $q$, in the domain of $f$ such that $f(q) \neq q$ and $f^k(q) = q$.

Proof: Let $O(x)$ be a periodic orbit of $f$ with odd period $n$. If $\omega^D < \omega^U$, then by Lemma 3.2.1c we already have the existence of all periods and we are done. So, suppose that $\omega^U < \omega^D$.

Since the period of our orbit is odd, one of the sets, $\tilde{U}_x$ or $\tilde{D}_x$ (which partition $O(x)$), must have more elements. Without loss of generality suppose that it is $\tilde{U}_x$—the proof for supposing the opposite is identical except for the direction of some of the inequalities. Since $f$, restricted to the orbit, is one-to-one, we must have an element of $\tilde{U}_x$ that maps to another element in this set, in other words, we have $\{a, f(a)\} \subset \tilde{U}_x$.

We claim that there exits points $b$ and $c$ in $O(x)$ such that $f^m(c) \leq a < f(a) \leq b < c \leq f(b)$ and $f(c) < f(a)$ for some $m, 1 \leq m \leq n - 2$.

To prove the claim, we first select $b$ in the orbit such that we minimize $q$ in the expression $f^q(b) \leq a < f(a) \leq b < f(b)$. The existence of such a $b$ is obvious, just take $b = f(a)$, and we get $f^{q+1}(a) \leq a < f(a) < f^2(a)$. Recall that $f(a) \in \tilde{U}_x$, so clearly $f(a) < f^2(a)$. Notice also that $3 \leq q + 1 \leq n$, so $2 \leq q \leq n - 1$.

Now, suppose that $f(b) < f^2(b)$. Let $c$ be a member of the orbit such that $c = f(b)$, then we have $f^{q+1}(c) \leq a < f(a) < c < f(c)$, but this gives us a number $(q - 1)$ that is less that $q$ and satisfies our construction which contradicts the minimality of $q$. So, $f(b) > f^2(b)$.

Thus, $f(b) \in \tilde{D}_x$. Let $l \geq 2$ be the smallest positive integer such that $f^l(b) \in \tilde{U}_x$. Notice that $f^{l-1}(b) \in \tilde{D}_x$ and $f^{l-1}(b) > f^l(b) < f^{l+1}(b)$. By the minimality of $l$ we have...
\{f(b), \ldots, f^{l-1}(b)\} \subset \tilde{D}_x$. Since \( b \in \tilde{U}_x \), \( f^{l-1}(b) \in \tilde{D}_x \), and \( \omega^U < \omega^D \) we get \( b < f^{l-1}(b) \).

Since \( \{f(b), \ldots, f^{l-1}(b)\} \subset \tilde{D}_x \), we have \( f(b) \geq \ldots \geq f^{l-1}(b) \), giving us \( b < f^{l-1}(b) \leq f(b) \).

Let \( c = f^{l-1}(b) \), which gives us \( b < c \leq f(b) \), and note that since \( b < f^{l-1}(b) = c \), \( a \neq b \neq c \). Since \( f(f^{l-1}(b)) = f^l(b) \in \tilde{U}_x \) we have \( f(c) \in \tilde{U}_x \). This fact implies that \( f(c) < f^2(c) \). Substituting \( c \), we have that \( f^q(b) = f^{q-l}(f^{l-1}(b)) = f^{q-l}(f(c)) \leq a \).

Assume for a moment that \( f(a) < f(c) \), then we have the following:

\[ f^{q-l}(f(c)) \leq a < f(a) < f(c) < f^2(c). \]

Let \( d = f(c) \), then \( f^{q-l}(d) \leq a < f(a) < d < f(d) \), but \( q - l < q \), which contradicts the minimality of \( q \). So, \( f(a) > f(c) \).

Thus we have \( f^{q-l}(f(c)) \leq a < f(a) \leq b < c \leq f(b) \) and \( f(c) < f(a) \). Let \( m = q - l + 1 \).

Since \( q \leq n - 1 \) implies \( q - l + 1 \leq n - l \), we get the following result: \( m \leq n - l \). Noting that \( q - l > 0 \) implies that \( q - l + 1 \geq 1 \), and that \( n - l \leq n - 2 \), we get that \( 1 \leq m \leq n - 2 \). Hence our claim is proved and we can make the construction.

Pick \( a, b, \) and \( c \), members of the orbit such that \( f^m(c) \leq a < f(a) \leq b < c \leq f(b) \), \( f(c) < f(a) \), and \( 1 \leq m \leq n - 2 \). Now, we claim that if \( p = m + 3 \) there exists a point, \( d \), in the domain of \( f \) such that \( f^p(d) \leq a < f(a) < f^2(d) < d < f(d) < f^3(d) \), \( 4 \leq p \leq n + 1 \), and \( f(y) > f^2(d) \) for all \( y \in [f^2(d), f(d)] \).

We prove the claim by first noting that since \( 1 \leq m \leq n - 2 \) implies that \( 4 \leq m + 3 \leq n + 1 \), we immediately get that \( 4 \leq p \leq n + 1 \).

Next, we take the facts that \( f(b) \geq c \) and \( f(c) < f(a) < b \) and discover that \( f([b, c]) > [b, c] \). So, there exists a point \( \alpha \in [b, c] \) such that \( f(\alpha) = c \). Thus \( f([\alpha, c]) > [\alpha, c] \).
which implies that there exists a point in \([\alpha, c]\) that maps to \(\alpha\). Let \(\beta\) be the smallest such point in \([\alpha, c]\). Note that for all \(y \in [\alpha, \beta]\), \(f(y) > \alpha\) by the minimality of \(\beta\). Also notice that \(f([\alpha, \beta]) \supseteq [\alpha, \beta]\), thus there exists a point \(d \in [\alpha, \beta]\) such that \(f(d) = \beta\). Note that \(\alpha \neq \beta \neq d\), because otherwise we would have a fixed point on a non-fixed orbit.

Now we have \(f^m(c) \leq a < f(a) \leq \alpha < d < \beta < c\). When we substitute the orbit of \(d\), we get \(f^p(d) \leq a < f(a) \leq f^2(d) < d < f(d) < f^3(d)\), noting \(f^m(c) = f^{p-3}(c) = f^{p-3}(f^3(d)) = f^p(d)\).

Observe also that since for all \(y \in [\alpha, \beta]\), \(f(y) > \alpha\), and that \(\alpha = f^2(d)\) and that \(\beta = f(d)\), we get that \(f(y) > f^2(d)\) for all \(y \in [f^2(d), f(d)]\). Thus our claim is proved.

Next, using the facts that \(f^2(d) = \alpha, \ d > \alpha, \ f^2(f(d)) = c\), and \(f(d) = \beta < c\), we get that \(f^2([d, \beta]) \supseteq [d, \beta]\), which implies that there exists at least one fixed point under \(f^2\) in \((d, \beta)\). Let \(\omega_1\) be the smallest of these and \(\omega_2\) be the biggest (these two points are not necessarily unique). So we now have \(f^p(d) \leq a < f(a) \leq f^2(d) < d < \omega_1 < \omega_2 < f(d) < f^3(d)\).

First notice that under \(f^2\) there are no fixed points in either \([d, \omega_1)\) or \((\omega_2, f(d)]\). So, by Property 3.2.1, since \(d \in \tilde{U}\), we get \([d, \omega_1) \subseteq \tilde{U}\); similarly since \(f(d) \in \tilde{D}\), we have \((\omega_2, f(d)] \subseteq \tilde{D}\). Thus for all \(y \in [d, \omega_1), y < f(y)\). Since \(f^2([d, \omega_1]) \supseteq [d, \omega_1), every point in [d, \omega_1)\) has a pre-image in \([d, \omega_1)\) under \(f^2\). Similarly, for all \(y \in (\omega_2, f(d)]\), \(y > f(y)\), and since \(f^2((\omega_2, f(d)]) \supseteq (\omega_2, f(d)]\), every point in \((\omega_2, f(d)]\) has a pre-image in \((\omega_2, f(d)]\) under \(f^2\).

So, we may make the following construction:

\[
d = \alpha_0 < \alpha_1 < \ldots < \alpha_i < \ldots < \omega_1 < \omega_2 < \ldots < \beta_i < \ldots < \beta_1 < \beta_0 = f(d),
\]

an infinite series of \(\alpha_i\)'s and \(\beta_i\)'s such that \(f^2(\alpha_i) = \alpha_{i-1}\) and \(f^2(\beta_i) = \beta_{i-1}\) with \(\alpha_{-1} = f^2(d)\) and \(\beta_{-1} = f^3(d)\).
Recall that \( a < f(a) \), \( f^2(a) < f^3(a) \), and \([f(a), f^3(d)] \supset [d, f(d)]\). So, \( f([a, f^2(d)]) \supset [d, f(d)]\), and thus the point(s) \( \omega_1 \) and \( \omega_2 \) have a pre-image in \([a, f^2(d)]\). Let \( \lambda_1 \) and \( \lambda_2 \) be those points in \([a, f^2(d)]\) with \( f(\lambda_1) = \omega_1 \) and \( f(\lambda_2) = \omega_2 \). Now, \( f(a) < d \), so \( f([a, \lambda_1]) \supset [d, \omega_1] \) which implies that for all \( \alpha_i \in [d, \omega_1] \) there exists a point \( \sigma_i \in [a, \lambda_1] \) such that \( f(\sigma_i) = \alpha_i \). Similarly, \( f(f^2(d)) = f^3(d) > f(d) \), thus \( f((\lambda_1, f^2(d)]) \supset [\omega_2, f(d)]) \), which implies that for all \( \beta_i \in (\omega_2, f(d)] \) there exists a point \( \tau_i \in (\lambda_2, f(d)] \) such that \( f(\tau_i) = \beta_i \).

Next we claim that for all \( k \geq p - 2 \), there is a point, \( \varrho \in (a, f^2(d)] \) such that \( f(\varrho) \neq \varrho \) and \( f^k(\varrho) = \varrho \). Once we have proven this claim, we have proven our lemma.

To prove our claim, we begin by looking at where our \( \tau \)'s and \( \sigma \)'s map to. Recall that \( f(\tau_{-1}) = \beta_{-1} = f^3(d) \). So, \( f^p(d) = f^{p-3}(f^3(d)) = f^{p-3}(f(\tau_{-1})) = f^{p-2}(\tau_{-1}) \). Similarly, for all \( i \geq 0 \), \( f(\tau_i) = \beta_i = f^{-2i}(\beta_{-1}) = f^{-2i}(f(\tau_{-1})) = f^{-2i}(f(d)) = f^{-2i+1}(d) \). So, \( f^p(d) = f^{p-(2i+1)}(f^{-2i+1}(d)) = f^p \cdot 2i-1(f(\tau_i)) = f^p \cdot 2i-1 \cdot 1(\tau_i) = f^p \cdot 2i(\tau_i) \).

Next we look at the \( \sigma \)'s.

Note \( f(\sigma_{-1}) = \alpha_{-1} = f^2(d) \). So, \( f^p(d) = f^{p-2}(f^2(d)) = f^{p-2}(f(\sigma_{-1})) = f^{p-1}(\sigma_{-1}) \). Similarly, for all \( i \geq 0 \), \( f(\sigma_i) = \alpha_i = f^{-2i}(\alpha_{-1}) = f^{-2i}(f(\sigma_{-1})) = f^{-2i}(f(d)) = f^{-2i+1}(d) \). So, \( f^p(d) = f^p \cdot 2i(f^{-2i+1}(d)) = f^p \cdot 2i(f(\sigma_i)) = f^p \cdot 2i \cdot 1(\sigma_i) \).

Now, look at all \( k \geq p - 2 \). Observe that \( k \) can have one of two forms: \( p \) plus an odd number or \( p \) plus an even number.
Case 1: let \( k = p + 2i, \ i \geq -1 \). For this case look at the interval \( (\lambda_2, \tau_i) \). By our construction
\[
 f((\lambda_2, \tau_i)) \cap (\lambda_2, \tau_i) = \emptyset, \ \text{yet} \ f^k((\lambda_2, \tau_i)) = f^{p+2i}((\lambda_2, \tau_i)) \supseteq (\lambda_2, \tau_i),
\]
so there exists a point \( q \in (\lambda_2, \tau_i) \) such that \( f(q) \neq q \) and \( f^k(q) = q \).

Case 2: let \( k = p + 2i + 1, \ i \geq -1 \). In this case look at the interval \( (\sigma_i, \lambda_1) \). Again, by our construction \( f((\sigma_i, \lambda_1)) \cap (\sigma_i, \lambda_1) = \emptyset, \ \text{yet} \ f^k((\sigma_i, \lambda_1)) = f^{p+2i}((\sigma_i, \lambda_1)) \supseteq (\sigma_i, \lambda_1), \) so there exists a point \( q \in (\sigma_i, \lambda_1) \) such that \( f(q) \neq q \) and \( f^k(q) = q \), which finishes this case and proves our claim and the lemma.

Let us summarize this lemma’s results. Let \( f \) be a function with an orbit, \( O(x) \), of odd period of length \( n \). If there exists \( q \), the smallest positive integer such that \( f^q \leq a < f(a) \leq b < f(b) \) for some \( a, b \) in \( O(x) \), then our lemma gives us \( p \leq q + 2 \leq n + 1 \). So, \( p - 2 \leq q \), thus for any \( k \geq q \), there exists a point, \( q \), such that \( f(q) \neq q \) and \( f^k(q) = q \).

Now we are ready to prove Štefan’s Theorem.

**Theorem 3.2.1 (Štefan’s Theorem):** Let \( O(x) \) be the minimal periodic orbit of \( f \) under the Šarkovskii ordering, i.e., the left-most in the ordering. Also, let \( O(x) \) have a length \( n \), odd and greater than one. Let the members of the orbit be labeled as \( y_1 < y_2 < \ldots < y_n \) and let \( t = (n + 1)/2 \).

Then one of the following cases holds.

Case A: \( f(y_{t-i}) = y_{t+i} \) for \( i = 1, \ldots, t - 1 \),
\[
 f(y_{t-i}) = y_{t+i-1} \text{ for } i = 0, \ldots, t - 2, \text{ and } f(y_n) = y_t,
\]
i.e., \( f^{n-1}(x) < \ldots < f^2(x) < f(x) < f^3(x) < \ldots < f^n(x) = x \).
Case B: \( f(y_{t-i}) = y_{t+i+1} \) for \( i = 0, \ldots, t - 2 \),
\[
f(y_{t+i}) = y_{t-i} \quad \text{for} \quad i = 1, \ldots, t - 1,
\]
i.e., \( x = f^n(x) < \cdots < f^3(x) < f(x) < f^2(x) < \cdots < f^{n-1}(x) \).

**Proof:** We will prove case B, but case A is proved in an identical manner. Let \( O(x) \) be a minimal orbit of period \( n = 2k + 1 \). Since we are actually only concerned with how the function orders the members of this orbit, we will look at an analogous structure: \( \hat{O}(x) = \{1, 2, \ldots, n\} \), an orbit of \( \hat{f} \), where \( \hat{f}(i) = j \) if \( f(y_i) = y_j \).

Since for a period of 3 this theorem is trivially true, we may assume that \( n \geq 5 \). Since \( f \) does not have orbits of all possible periods, we can deduce from Lemma 3.2.1c that \( \omega^U < \omega^D \). Without loss of generality we may assume that \( \hat{U}_x \) has more elements than \( \hat{D}_x \). This assumption leads to case B; if \( \hat{D}_x \) has more elements, then the proof gives case A. Since \( \hat{U}_x \) has more elements than \( \hat{D}_x \), there is a member of \( \hat{O}(x) \) such that \( \{a, \hat{f}(a)\} \subset \hat{U}_x \) giving us \( a < \hat{f}(a) < \hat{f}^2(a) \). So, we have \( 1 = \omega_{\text{min}} \leq a < \hat{f}(a) < \omega^U < \omega^D = \omega^U + 1 \leq \omega_{\text{max}} = n \).

Assume that \( a \neq \omega_{\text{min}} \). Then there exists a positive integer, \( s \), such that \( \hat{f}^s(a) = \omega_{\text{min}} \), for some \( s \leq n - 1 \) (since \( \hat{f}^n(a) = a \)). Note that \( s \geq 3 \) since \( n \geq 5 \); thus, \( 3 \leq s \leq n - 1 \). Now, we wish to apply Lemma 3.2.1d, so we need to know the range of possible values for the \( q \) of that lemma. For an upper bound we need to look at the value of \( q \) that results from choosing a valid \( b \) closest to \( a \) in the orbit path: \( \hat{f}^q(b) \leq a < \hat{f}(a) = b < \hat{f}(b) \). This fact implies \( \hat{f}^{q+1}(a) \leq a < \hat{f}(a) < \hat{f}^2(a) \), which implies \( q + 1 \leq s \leq n - 1 \), further implying \( q \leq s - 1 \leq n - 2 \). So, by Lemma 3.2.1d we pick a \( k \geq q \), say, \( k = n - 2 \), and we have \( \hat{f}^n(q) = q \) and \( \hat{f}(q) \neq q \). Since \( n - 2 \) is odd, we have the existence of an orbit of odd period less than \( n \), contradicting the minimality of \( n \). Thus we must have \( a = \omega_{\text{min}} = 1 \).
Recall that \( a \) was chosen as a member of \( \tilde{U}_x \) with the property that it is mapped to another element of this set. The lemma has just shown that \( a \) is \( \omega_{\text{min}} \). Since there is only one element, \( \omega_{\text{min}} \), in \( \tilde{U}_x \), \( a \) must be unique, and every other element of \( \tilde{U}_x \) must map to \( \tilde{D}_x \). Since the function is one-to-one on the orbit, and since there are more elements in \( \tilde{U}_x \) than in \( \tilde{D}_x \), we must conclude that the difference in number of elements is one. We also conclude that, with the exception of \( \omega_{\text{min}} \), every element in one set maps to an element in the other set. Thus, \( \omega_U = k + 1 \) and \( \omega_D = k + 2 \) (\( k \) from \( n = 2k + 1 \)).

Next we claim that actually \( \omega_U = \hat{f}(a) \).

Suppose that the claim is not true; then \( \hat{f}(a) < \omega_U \). This fact gives us \( \hat{f}^s(\omega_U) = a < \hat{f}(a) < \omega_U < \hat{f}(\omega_U) \). Notice that \( s \leq n - 2 \), since \( \hat{f}^n(\omega_U) = \omega_U \), \( a \neq \omega_U \), and \( \hat{f}(a) \neq \omega_U \). Again we wish to use Lemma 3.2.1d and need to find an upper bound to \( q \). Notice that \( q \) cannot be larger than \( s \). Using Lemma 3.2.1d, we note that we can choose \( k = n - 2 \), contradicting the minimality of \( \hat{O}(x) \). Thus, \( \hat{f}(a) = \omega_U = k + 1 \).

Now we will show that \( \hat{f}(a) = \omega_D \).

As before, we assume that \( \hat{f}(a) \neq \omega_D \). Note that \( \hat{f}(\omega_D) < \omega_U \) since otherwise our orbit would have a period of two. Therefore, we have \( \omega_{\text{min}} = a \leq \hat{f}(\omega_D) < \hat{f}(a) = \omega_U < \omega_D < \hat{f}^2(a) \). Obviously there is an \( s \) such that \( \hat{f}^s(\omega_D) \leq a \), and again the \( q \) of Lemma 3.2.1d cannot be larger than \( s \). Note that \( s \leq n - 3 \) because \( \hat{f}(a) \neq \omega_D \) and \( \hat{f}^2(a) \neq \omega_D \). Thus using Lemma 3.2.1d a third time with our choice of \( k \) being \( k = n - 2 \), we arrive at a contradiction to the minimality of \( \hat{O}(x) \).

Thus \( \hat{f}^2(a) = \omega_D = k + 2 \). Note that \( \hat{f}^3(a) \neq \hat{f}(a) \) because \( \hat{f}^3(a) < \hat{f}^2(a) \) and \( \hat{f}^3(a) \neq \hat{f}(a) \). Now we have \( 1 = \omega_{\text{min}} = a = \hat{f}^n(a) < \ldots < \hat{f}^3(a) < \ldots < \hat{f}(a) < \hat{f}^2(a) < \ldots < \omega_{\text{max}} = n = 2k + 1 \).
We also have \( \hat{f}^{2s+1}(a) < \hat{f}(a) \) and \( \hat{f}^{2s+2}(a) < \hat{f}^{2}(a) \) for \( 1 \leq s \leq k - 1 \) because, with the exception of \( a \), every element in \( \mathcal{U}_x \) maps to \( \mathcal{D}_x \) and every element in the latter set maps to the former set.

All that remains of the proof is to show that the sequences \((a, \hat{f}^{2}(a), \hat{f}^{4}(a), \ldots)\) and \((\hat{f}(a), \hat{f}^{3}(a), \hat{f}^{5}(a), \ldots)\) are monotonic.

Since we know that \( a < \hat{f}^{3}(a) < \hat{f}(a) < \hat{f}^{2}(a) < \hat{f}^{4}(a) = \omega_{\text{max}} \), we have proven the theorem for \( n = 5 \). We will use this fact and the following lemma to complete the proof of the theorem.

**Lemma 3.2.1e:** If \( y < \hat{f}^{2}(y) \) and \( \hat{f}^{4}(y) < \hat{f}^{2}(y) \) for some \( y \) in \( \hat{O}(x) \), then \( \hat{f}^{4}(y) < \hat{f}(a) < \hat{f}^{2}(a) < \hat{f}^{2}(y) \).

If one were proving case B, then one would need the alternate lemma: if \( y > \hat{f}^{2}(y) \) and \( \hat{f}^{4}(y) > \hat{f}^{2}(y) \) for some \( y \) in \( \hat{O}(x) \), then \( \hat{f}^{4}(y) > \hat{f}(a) > \hat{f}^{2}(a) > \hat{f}^{2}(y) \).

**Proof:** Note that \( \hat{f}^{n-2}(\hat{f}^{2}(x)) = x < \hat{f}^{2}(x) \) and \( \hat{f}^{n-2}(\hat{f}^{4}(x)) = \hat{f}^{2}(x) > \hat{f}^{4}(x) \). So, \( \hat{f}^{n-2}(\hat{f}^{2}(x), \hat{f}^{4}(x))) > (\hat{f}^{2}(x), \hat{f}^{4}(x)) \). Thus, there is a point \( z \in (\hat{f}^{2}(x), \hat{f}^{4}(x)) \). By the minimality of \( \hat{O}(x) \), \( z \) must be fixed by \( \hat{f} \), otherwise we would get an odd period to the left of \( n \) in the \( \text{Š} \)arkovskii ordering. However, by Lemma 3.2.1b, if such a fixed point is not between \( \omega^U \) and \( \omega^D \) we get the existence of all periods. Since period 3 does not exist, we know that \( \omega^U < z < \omega^D \), but we have already shown that \( \omega^U = \hat{f}(a) \) and \( \omega^D = \hat{f}^{2}(a) \). Recall that \( y \) was chosen to be on the same orbit as \( a \), thus \( \hat{f}^{4}(y) \geq \hat{f}(a) > \hat{f}^{2}(a) > \hat{f}^{2}(y) \), and we complete the proof.

Now, let \( \hat{f}^{+}(a) < \hat{f}^{+2}(a) \). Assume that \( \hat{f}^{+4}(a) < \hat{f}^{+2}(a) \). By Lemma 3.2.1e, we have \( \hat{f}^{+4}(a) > \hat{f}(a) > \hat{f}^{2}(a) > \hat{f}^{+2}(a) \).
Suppose \( t \) is even, then \( t + 4 \) is even. So we have \( f^{t+4}(a) < f^2(a) \), which contradicts the fact we have already proven: \( f^{2s+2}(a) > f^2(a) \). Similarly, suppose \( t \) is odd, then \( t + 2 \) is odd. Then we have \( f^{t+2}(a) > f(a) \), which contradicts the fact we have already proven: \( f^{2s+1}(a) < f(a) \).

Thus, our assumption is wrong. Further note that \( f^{t+4}(a) \neq f^{t+2}(a) \), since equality would only be possible if we were dealing with an orbit of period 1 or 2, and we are dealing with an odd period greater than 3. So, we have that if \( f'(a) < f^{t+2}(a) \), then \( f^{t+2}(a) < f^{t+4}(a) \). We know that \( a < f(a) < f^2(a) < f^4(a) \). So for \( t \) even, we have \( f^2(a) < f^{2+2}(a) \) and by induction we get \( f^{2i}(a) < f^{2i+2}(a) \) for all \( i = 0, 1, 2, \ldots \). Similarly for \( t \) odd, we have \( f^1(a) > f^{1+2}(a) \) and by induction we get \( f^{2i+1}(a) > f^{2i+3}(a) \) for all \( i = 0, 1, 2, \ldots \). So the sequences are monotonic, and the theorem is proved.

3.3 Formation of periodic points and the Kneading Sequence

Now that we have looked at the formation of periodic points in general by Šarkovskii’s theorem and noted some of their structure in Štefan’s theorem, we will look at an example in detail. Our example studies a subclass of the transitional families, those transitional families whose first derivative is monotonic on the intervals \([0, c]\) and \([c, 1]\).

These families of functions are particularly amenable to study through kneading theory. We will discover a surprising property that this class of transitional families has, namely that the critical points of these systems traces all bifurcations and thus the formation of periodic points follow the kneading sequence.

This exploration will take several steps. First, we will show how many periodic points of each length there are, and prove that there is a one-to-one correspondence between periodic points and periodic itineraries. Secondly, we will show that each orbit (for period-doubling bifurcations)
and one of each orbit pair (for saddle-node bifurcations) pass through the critical point, c, thus taking on the kneading sequence. By using the ordering of the itineraries, we will show that all orbits or orbit-pairs come into existence one at a time.

To begin our discussion of periodic orbits and itineraries we will first examine periodic itineraries. We will define an n-itinerary as a periodic itinerary that repeats after every n entries.

**Definition 3.3.1:** An n-itinerary has the form \( \overline{s_1s_2s_3 \ldots s_n \ldots} \), which is written \( \langle s_1 s_2 s_3 \ldots s_n \rangle \) for shorthand.

Periodic itineraries will be written without ellipsis marks or the over bar, the endless repetition being understood.

Both itineraries and orbits can be "viewed" in different frames of reference. For example, a point with a period of 2 can also be looked at as having a period of 4, 6, or any other even length. To eliminate the ambiguity, recall that we defined a point's period to be the smallest of these frames of reference. Similarly these orbits could be viewed as a 2-, 4-, or 6-itinerary (or any 2n-itinerary), for example: \( (10) \), \( (1010) \), \( (101010) \). Also note that all of these itineraries represent the same period-two orbit of \( f \), but as it is manifested as a fixed point under \( f^2 \), \( f^4 \), and \( f^6 \), respectively.

Before we start looking at the formation of periodic points, we will show exactly how many of each kind of period the system has at \( \lambda_1 \). We will show that for every periodic itinerary generated by the kneading theory, there is exactly one periodic orbit whose points follow that sequence. In general this is not necessarily the case because while a point with a periodic orbit must have a periodic itinerary (though it may be degenerate), a point with a periodic itinerary does not necessarily have a periodic orbit. First, we will prove a theorem that shows how many fixed points \( f_{\lambda_1}^n \) has. Then by noting that there are exactly the same number of n-itineraries as there are fixed points of \( f_{\lambda_1}^n \).
and that each periodic point can have only one itinerary, we will conclude our one-to-one correspondence.

**Theorem 3.3.1:** Any member of the transitional family, $f_{e_i}^n$, whose first derivative is monotonic on the intervals $[0, c]$ and $[c, 1]$, has exactly $2^n$ fixed points.

**Proof:** This proof will be taken in several steps and lemmas. The basic idea will be to count the fixed points by first relating them to the critical points of the function, and then counting how many critical points there are.

**Lemma 3.3.1a:** The function, $f_{e_i}^n$, whose first derivative is monotonic on the intervals $[0, c]$ and $[c, 1]$, has one more fixed points than it does critical points.

**Proof:** Let the $c_{ni}$'s be the critical points of $f_{e_i}^n$ indexed by $i = 1 \ldots j$. For the rest of the proof of this theorem and its associated lemmas, we will denote $f_{e_i}$ as $f$.

The orbits of these critical points must pass through $c$ within $n - 1$ iterations. For if $c_{ni}$ is a critical point, then $(f^n)(c_{ni}) = 0$, but by the chain rule

$$(f^n)'(c_{ni}) = f'(f^{n-1}(c_{ni})) \cdot f'(f^{n-2}(c_{ni})) \cdot \ldots \cdot f'(f(c_{ni})) \cdot f'(c_{ni}) = 0.$$  

Thus since $f'(x) = 0$ if and only if $x = c$, we can conclude that $c_{ni} = c$ or $f^k(c_{ni}) = c$ for some $k = 1 \ldots (n - 1)$.

Note: since every critical point of $f^n$ always passes through $c$, all the critical points of $f^n$ always lie on the same orbit, namely the orbit through $c$.

Notice also that $f(c) = 1$, $f(1) = 0$, and $f(0) = 0$. So, any $c_{ni}$ passing through $c$ within $n - 2$ iterations passes on to 0 (i.e. $f^n(c_{ni}) = 0$). Those $c_{ni}$ passing through $c$ on the $(n - 1)^{th}$ iteration passes to 1, i.e., $f^n(c_{ni}) = 1$.  

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Thus $f^n(c_n) \in \{0,1\} \forall c_n$. Since $\forall x \in [0,1], f^n(x) \in [0,1]$, we can see that all the critical points must be relative extrema at $\lambda_i$. (Shortly we will prove that this is true for $\lambda_0 < \lambda \leq \lambda_1$.) Additionally, since all relative minima have a value of 0 and all relative maxima have a value of 1, between every max-min pair, $f^n(x)$ must intersect the line $y = x$, and thus create a fixed point. So, $f^n(x)$ has $j - 1$ fixed points sitting between its $j$ critical points. Finally, since $f^n(0) = f^n(1) = 0$, the graph of $f^n(x)$ intersects the line $y = x$ an additional two times—one of those times at the lower endpoint $f^n(0) = 0$. Since the function is one whose first derivative is monotonic on the interval $[0,c]$, the graph must rise from 0 to its relative maximum without crossing the line $y = x$. The endpoints are not critical points, because if they were, they would then have to pass through $c$, but $f^n(0) = f^n(1) = 0$ for all $n$. Therefore $f^n(x)$ has $(j - 1) + 2$ fixed points, which is one more than its number of critical points, and our lemma is proved.

Lemma 3.3.1b: The function, $f^n_{\lambda_1}$, whose first derivative is monotonic on the intervals $[0,c]$ and $[c,1]$, has exactly $2^n - 1$ critical points.

Proof: First, all $c_n$ are on an orbit passing through $c$, and the orbit of $c$ is eventually fixed at zero. The orbit of $c$ is not periodic nor wandering. So, under backward iteration of $c$ there can be no periodic orbits. Now, except for 1 and 0, all points under backwards iteration come from two distinct points, because $f$ maps $I$ onto itself twice. For $f^n(x)$ (recall it is understood that all functions are at $\lambda_1$), every member of the reverse orbit of $c$ up to $n - 1$ iterations corresponds to a critical point of $f^n(x)$. Thus, including $c$, we have $1 + 2 + 2^2 + \ldots + 2^n - 1 = \sum_{i=1}^{n} 2^{i-1}$ critical points; that is exactly $2^n - 1$ critical points if all these critical points are unique. Note that there can be no more critical points as they would not pass through $c$ within $n - 1$ iterations as required.
With careful inspection, it is clear that these \( c_{ni} \)'s are unique for a fixed \( n \). First, clearly every backwards iteration of a \( c_{ni} \) gives a unique pair (siblings) since they come from the disjoint sets, \((1,c)\) and \((c,0)\). Since \( f \) is unimodal, by inspection any element in \( f \)'s range (except for 1) comes from two distinct elements in the domain, one of these elements must be less than \( c \) and the other must be greater than \( c \). Any duplication among “cousins” (two points generated in the same generation from different parents, say \( c_{n3} \) and \( c_{n5} \) of Figure 4) will cause, under forward iteration, a sibling match (in our example \( c_{n1} \) and \( c_{n2} \)), which cannot happen. Finally, there cannot be a duplication in different generations (say \( c_{n3} \) and \( c_{n12} \) of Figure 4) because under forward iteration they must become \( c \) in the same number of steps, causing \( c \) to exist twice in the tree which is a contradiction to the fact that \( c \) maps to 1 (which maps to 0 and stays at 0 forever). Thus, at \( \lambda = \lambda_1, f^n(x) \) has exactly \( 2^n - 1 \) critical points, and we have proved the lemma.

By putting the two lemmas together, we prove our theorem that \( f_{\lambda_1}^n \) has exactly \( 2^n \) fixed points, and we complete the theorem.

Now, we will prove the one-to-one correspondence.

**Theorem 3.3.2:** There exists a one-to-one correspondence between the periodic itineraries and the periodic orbits of \( f_{\lambda_1}^n \), a transitional family whose first derivative is monotonic on the intervals \([0,c]\) and \([c,1]\).
Proof: Recall that the fixed points of $f^n_A$ are those points which, under $f$, have a period of length 1, $n$, or a factor of $n$. Any fixed point of $f^n_A$ can be represented by an $n$-itinerary. For a given $n$, the maximum number of unique $n$-itineraries is $2^n$, exactly the number of fixed points. Recall that entries are strings of 1's and 0's; $c$'s are not possible since $c$ lies on an eventually fixed orbit, not on a periodic orbit. Note that any point can only have one itinerary, because a given point has only one orbit and any given orbit can yield only one itinerary. Since the fixed points have only one itinerary of $f^n_A$, there must be a one-to-one correspondence between these fixed points and the $n$-itineraries of $f^n_A$.

The preceding argument gives us two facts: the number of fixed points and the correspondence between them and the itineraries. We also determined that each $f^n$ had exactly $2^n - 1$ relative extrema at $A_0$. We will now prove a stronger statement in the following theorem:

**Theorem 3.3.3:** All the critical points of $f^n_A$ for $\lambda_0 < \lambda \leq \lambda_1$ are relative extrema.

**Proof:** We will prove this fact by using the chain rule. Fix $\lambda$, and assume that $n$ and a critical point of $f^n$, say $c_k$, are given. By the chain rule

$$(f^n)'(c_k) = f'(f^{n-1}(c_k)) \cdot f'(f^{n-2}(c_k)) \cdot \ldots \cdot f'(f(c_k)) \cdot f'(c_k) = 0.$$ 

As we have shown, one or more of these $f'(c_k)$ must be $c$, $0 \leq i \leq n - 1$. Let $i = 1$ be the smallest iteration that $f'(c_k) = c$. Now we can rewrite the chain rule with a different frame of reference:

$$(f^n)'(c_k) = f'(f^{n-1-l}(c)) \cdot f'(f^{n-2-l}(c)) \cdot \ldots \cdot f'(c) \cdot f'(x_{i-1}) \cdot \ldots \cdot f'(x_2) \cdot f'(x_1) = 0$$

where $x_i = f^i(c_k), 1 \leq i \leq l - 1$. Notice that by the minimal choice for $l$, none of the $x_i$'s are $c$, but note that $x_1 = c_k$. 

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Next, we carefully choose two points on either side of $c_k$ and show that their slopes have opposite signs. Since all the $x_i$'s are not $c$, there exists open intervals around each one such that $f$ is monotonic. Let $J_1$ be an open interval around $x_1$ such that $f$ is monotonic on $J_1$. Then $f$ restricted to this interval is a one-to-one function. Also note that since $x_1$ is in $J_1$, $x_2$ (which is $f(x_1)$) must be contained in $f(J_1)$. Since $x_2$ is not $c$, there exists an open interval around $x_2$ in which $f$ is monotonic. Let $J_2$ be the intersection of that interval and $f(J_1)$.

We continue this process until we get to $J_{l-1}$. Now, $f(J_{l-1})$ maps to an interval that contains $c$, since $J_{l-1}$ contains $x_{l-1}$ and $f(x_{l-1}) = c$. We will let $J_l = f(J_{l-1})$. Since $f$ is unimodal, $c$ is a maximum. So, there exists two points, $p$ and $\hat{p}$, in $J_l$ on either side of $c$. As previously defined, $f(p) = f(\hat{p})$. Notice that $f^{-1}(p)$ and $f^{-1}(\hat{p})$ lie on either side of $x_{l-1}$ and within $J_{l-1}$. Similarly, by the way we constructed the $J_i$'s, and since $f$ is monotonic on all these intervals, $f^{-(l-1)}(p)$ and $f^{-(l-1)}(\hat{p})$ lie on either side of $x_1$ and are within $J_1$. Let $q = f^{-(l-1)}(p)$ and $\hat{q} = f^{-(l-1)}(\hat{p})$.

Consider the following:

\[
(f^n)'(q) = f'(f^{n-1}(q)) \cdot f'(f^{n-2}(q)) \cdots f'(f^1(q)) \cdot f'(f^0(q)) \cdot f'(q)
\]

\[
(f^n)'(\hat{q}) = f'(f^{n-1}(\hat{q})) \cdot f'(f^{n-2}(\hat{q})) \cdots f'(f^1(\hat{q})) \cdot f'(f^0(\hat{q})) \cdot f'(\hat{q})
\]

and notice that they have the opposite sign.

Note that the slopes of $f'(q)$ and $f'(\hat{q})$ have the same sign for all $0 \leq i < l$. Furthermore, $f^i(q) = f^i(\hat{q})$ for all $i > l$, and thus they have the same slope. So, only the slopes of $f^l(q) = p$ and $f^l(\hat{q}) = \hat{p}$
have different signs, since they are on opposite sides of $c$. Thus $(f^n)(q)$ and $(f^n)(\dot{q})$ have opposite signs and thus the critical point $c_k$ of $f^n$ must be an extremum.

Thus we have proven the theorem.

Some of the itineraries represent orbits created by period-doubling bifurcations, the rest come from saddle-node bifurcations. We will examine the order that these types of bifurcations occur relative to each other in an attempt to understand how these transitional families reach their chaotic state. Since the itinerary and kneading sequence are vital to the following arguments, we will briefly digress from our main discussion and look at some of the properties of these sequences.

One helpful property of the itineraries is that, for a given $n$, it is easy to see which itineraries correspond to fixed points, period $n$ points, and factors-of-$n$ periodic points. Furthermore, it is also easy to see which of these itineraries lie on the same orbit.

For any positive integer, $n$, $f^n(x)$ has fixed points at the fixed points of $f$. There are two such points, represented by the $n$-itineraries $(000 \ldots 0)$ and $(111 \ldots 1)$. Similarly, any itinerary, which represents a point with a periodic orbit whose length is a factor of $n$, can be identified by the repeating pattern within the $n$-itinerary. Points that share the same orbit, have the same $n$-itinerary differing only in that their entries are left-shifted.
For example, there are sixteen possible n-itineraries when \( n = 4 \):

<table>
<thead>
<tr>
<th>Fixed</th>
<th>Period 2</th>
<th>Period 4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>P-dB Orbit</td>
<td>P-dB Orbit</td>
</tr>
<tr>
<td>0000</td>
<td>1010</td>
<td>1011</td>
</tr>
<tr>
<td>1111</td>
<td>0101</td>
<td>1101</td>
</tr>
<tr>
<td></td>
<td>1110</td>
<td>0010</td>
</tr>
<tr>
<td></td>
<td>0111</td>
<td>0001</td>
</tr>
</tbody>
</table>

In the table the itineraries are labeled according to their period and clustered in columns by common orbit. Within the orbit column the itineraries are listed from largest to smallest in the ordering that was introduced in Definition 2.7.1. In the chart P-dB signifies itineraries created in period-doubling bifurcations, and S-nB signifies itineraries created in saddle-node bifurcations. As one can see, the orbits of period 2 stand out because their itineraries consist of a repeating subsequence of two elements. Recall that the actual itineraries are infinitely repeating sequences; only the first repetition with its critical information is written.

All period-doubling bifurcations occur around a point whose period has length one half of the new period. So in our example, the period-four bifurcation must form around the points in the single period-two orbit. The period-doubling bifurcations are identifiable as a copy of the highest-valued parent itinerary with the last entry changed. In the example, \((1010)\) identifies the orbit around which a period-doubling bifurcation occurs. This bifurcation creates the orbit whose itinerary is \((1011)\). This process will be discussed in greater detail in the rest of this chapter.

The number of all other period-four itineraries must be even, since saddle-node bifurcations occur in pairs. Our example has only two remaining, the two saddle-node orbits, so these must be
formed from the same bifurcation. Lastly, notice that their itineraries differ only by one entry. We will now look at these ideas in more detail.

We claim that every bifurcation creates a point that becomes \( c \) for some single \( \lambda \), and at this \( \lambda \), the point’s itinerary is the kneading sequence. We will prove this claim in stages with the help of a few lemmas.

**Lemma 3.3.4a:** A bifurcation can never occur at \( c \).

**Proof:** Notice that \( f^1_\lambda \) has a critical point at \( c \), for all \( n \). We can see this fact by applying the chain rule to \((f^n)'(c) = f'(f^{n-1}(c)) \cdot f'(f^{n-2}(c)) \cdot \ldots \cdot f'(f(c)) \cdot f'(c) = 0 \). We get equality to 0 because \( f'(c) = 0 \). Now, recall that a bifurcation occurs in \( f^1_\lambda \) only when the slope of the graph is one. Since the slope of the graph is always 0 at \( c \) no matter what the value of \( n \), we can see that no bifurcation can occur here. \( \square \)

Although no bifurcation can occur at \( c \), we will show that every cluster of related bifurcations creates a point that will eventually become \( c \) for some value of \( \lambda \). We begin with the observation of the following property.

**Property 3.3.4b:** At any place in two itineraries where there is a discrepancy, the points of the orbits corresponding to the discrepancy must lie on opposite sides of \( c \).

Alternatively, we can say that if two points have the same itinerary, the corresponding elements in their two orbits are always on the same side of \( c \).

**Property 3.3.4c:** If a bifurcation occurs at \( \lambda_a \), then for \( \lambda \) sufficiently close to \( \lambda_a \) we have the following:

1. If the bifurcation is of the saddle-node type, the orbit-pair has the same itinerary and
(2) If the bifurcation is of the period-doubling type, the new orbit has the same itinerary as the orbit it formed around.

Essentially this observation follows from two previous facts: a bifurcation cannot occur at $c$ so there must be a small change in the value of $\lambda$ for which $c$ is not in between the orbit pair, in case (1), or orbit and the point it formed around, in case (2).

This property gives us the fact that points from different orbits share the same itineraries, while at $\lambda_1$ every periodic point has its own unique itinerary, since by Theorem 3.3.2 there exists a one-to-one correspondence between periodic orbits and periodic itineraries. By the first property we note that the only way an itinerary can change is if one of the points on the orbit changes from one side of $c$ to the other. Since our transitional families are continuous functions that depend smoothly on $\lambda$, the only way for a point to move from one side of $c$ to the other is for it to become $c$ for some $\lambda$. We will call this passing through $c$, the “maturing” of the orbit or itinerary.

**Definition 3.3.2:** An orbit *matures* as one of its members becomes $c$, and an orbit is called a *mature orbit* if its itinerary is the same one that it will possess when $\lambda = \lambda_1$.

If we are careful in defining what we mean by ‘bifurcation’, this maturing forces all bifurcations to happen in sequence.

Now, we will carefully define what one bifurcation means. Clearly more than one bifurcation must occur at the same time. For example, if a saddle-node bifurcation forms an orbit of period 3, three such bifurcations must occur simultaneously. Recalling that a saddle-node bifurcation creates two points on different orbits, the three simultaneous bifurcations form two orbits of period 3. The same can be said for period-doubling bifurcations. Suppose the system bifurcates in this fashion, forming two points of period 6 (these being on the same orbit). Since there must be six period 6
points, two other period-doubling bifurcations must have occurred at the same $\lambda$. We will state two definitions to clarify this situation.

In the following theorems and lemmas, a bifurcation will refer specifically to the family of related bifurcations that create periodic points all belonging to the same orbit. Also, a pair of unrelated bifurcations are two bifurcations that create points, which do not belong to the same orbit.

The related bifurcations also occur in an infinite number of $f^n$'s as well; if $f^n$ bifurcates, then so must $f^{kn}$ for any positive integer $k$. This infinitely large family of bifurcations all creates the same orbit and actually the same itinerary. The only difference is the point of reference, in other words, the fundamental cycle is repeated $k$ times in the $kn$-itinerary.

So we claim that the system bifurcates to create a single new orbit or orbit-pair for a given $\lambda$.

**Theorem 3.3.4:** No two unrelated bifurcations happen at the same value of $\lambda$.

**Proof:** We will look at this problem in three cases:

1. The system cannot suffer two simultaneous unrelated bifurcations of differing period, where neither of which are multiples of the other, at the same value of $\lambda$,

2. The system (and thus a given $f^n$) cannot suffer two simultaneous unrelated bifurcations of the same period per value of $\lambda$, and

3. The system cannot suffer two simultaneous unrelated bifurcations of differing period, where one is a multiple of the other, at the same value of $\lambda$.

Case 1: We show that we cannot have two unrelated bifurcations which form orbits of differing periods, neither of which are multiples of the other. First note that any simultaneous bifurcations of differing periods, where neither of which are multiples of the other, cannot occur at
the same point, since if this situation were to happen they will form the same orbit and a single orbit cannot have two different periods.

Now, we have only two cases of how these unrelated bifurcations can occur relative to one another and $c$: $c$ could be to one side of the two pair or it could be between the two pairs (recall that these two pairs are either two points one from each orbit pair of a saddle-node bifurcation or both of the same orbit of a period-doubling bifurcation). In either case both new pairs of points cannot have one of their members mature and allow the other pair to have a maturing point as well.

Figure 5 shows the case of $c$ in the middle by labeling the point pair from one bifurcation by

$$y = x \quad p_n, p'_{n}, c, p_m, p'_{m}$$

**Figure 5**

$p_n$, and $p'_{n}$, while the points from the second bifurcation are labeled $p_m$, and $p'_{m}$. The former point is the one that is closest to $c$, and thus will cross $c$. As can be seen from Figure 5, $p_n$ and $p_m$ cannot both end up on the side of $c$ opposite from the side they exist on now without crossing each other. Since they represent orbits of different periods, they cannot have the same $x$-value, since any point in the domain of the function can have a period of only one length.

The case of $c$ being to one side is similar to this case. Say, without loss of generality, $c$ was to the left of all the points. Now, $p_m$ would have to cross $p'_{n}$, since the latter cannot be on the left side of $c$ at $\lambda = \lambda_1$ for then its itinerary and the itinerary of $p_n$ would be the same in the saddle-node case. While in the period-doubling case, we note that the birthing point is trapped between $p_m$ and $p'_{m}$. If both these points cross $c$, so must the birthing point in between and then the
pair's orbit and the birthing point's orbit will share the same itinerary. Thus we have proved Case 1.

Case 2: We show that we cannot have $f^n$ suffer two simultaneous non-related bifurcations of the same period. If we can show that two points from unrelated bifurcations cannot cross, then by the previous argument we show that they must also occur one at a time (since the essence of the argument was that two points from unrelated bifurcations could not cross). Here is the place where we finally appeal to the Schwarzian condition; more precisely to the fact that if a function has a negative Schwarzian derivative then the function’s first derivative cannot have a positive relative minimum or a negative relative maximum. We will prove our case with the following lemmas and definition.

**Lemma 3.3.4d:** Let the Schwarzian derivative of $f$ be negative. If $f$ has four fixed points, $p_1 < p_2 < p_3 < p_4$, then there must be a critical point, $c$, of $f$ such that $p_1 < c < p_4$.

**Proof:** Let $f$ have a negative Schwarzian derivative, then by Property 2.4.1, $f'$ cannot have a positive relative minimum or a negative relative maximum. Let $f$ have four fixed points, $p_1 < p_2 < p_3 < p_4$, such that there is no critical point, $c$, of $f$ with $p_1 < c < p_4$. By the Mean Value Theorem, $f$ must have a slope of one in between each of these fixed points, since all the fixed points are on the line $y = x$. So, we have three points where the function has a slope of one, in between which the slope is never zero, since there is no critical points there.

By the continuity of $f$, we note that the slope of $f$ between these three points is never negative. Recall that the slope of the function is not constant, hence there must be a point, $p$, $p_2 < p < p_3$ where the slope is less than one. These facts give us the result that $f'$ must have a
positive relative minimum contradicting Property 2.4.1. Thus, if \( f \) has a negative Schwarzian derivative, it cannot have four or more fixed points without a critical point in between.

**Definition 3.3.3:** Two distinct fixed points, \( p_1 \) and \( p_2 \), at \( \lambda_a \) are said to fuse at \( \lambda_b > \lambda_a \) if \( p_1 \) and \( p_2 \) converge to the same fixed point as \( \lambda \) converges to \( \lambda_b \).

**Lemma 3.3.4:** Two points from unrelated bifurcations of \( f'' \) cannot fuse.

**Proof:** We will start our proof by commenting that if two fixed points of \( f'' \) were to fuse, the graph of the function at the point of fusion must have a slope of 1, i.e., be tangent to the line \( y = x \). By the Mean Value Theorem, there exists a point between the converging fixed points where the slope of the function is 1. This point exists no matter how close the two points get, so they must meet at that point. Since the function has a slope of one at the point of fusion, this act can never happen at a critical point, which always has a slope of 0. Hence the point of fusion must occur (to borrow a term from Milnor and Thurston [MT]) in a lap of the function, i.e., an open interval of the function that has critical points—or possibly an endpoint of the domain—at the boundary and only at the boundary.

One of two possibilities exist, the two unrelated bifurcations occurred in the same lap or they did not.

Case (a): Assume that the bifurcations occurred in the same lap. This situation places at least four fixed points in series without an intervening critical point and contradicts the Schwarzian condition.

Case (b): Assume that the bifurcations occurred in different laps, label the bifurcations A and B. First note that since all critical points are relative extrema, the sign of the slopes of any two adjacent laps must be opposite. Recall that bifurcations only occur when the slope of the graph is
one, thus bifurcations cannot occur in adjacent laps since in one of them the slope of all points is negative. Further, this implies that two bifurcations must occur with an odd number of laps between them. We will represent this situation by the following series of fixed points and critical points:

\[ c_{a1}, p_{a1}, p_a, c_{a2}, \ldots, c_{b1}, p_b, p_{b1}, c_{b2}. \]

One of two things can occur as our points move toward each other on their way to fusing: One of the points crosses an intervening critical point first, or some of the intervening critical points fuse together.

Case (b1): One of the points, say \( p_a \), crosses an intervening critical point \( c_{a2} \) before any intervening critical point have a chance to fuse. When \( p_a \) is \( c_{a2} \), all critical points of \( f^n \) lie on that periodic orbit, namely \( c_{b1} \) and \( c_{b2} \) lie on the same orbit as \( p_a \). These two points on \( p_a \)'s orbit must have appeared at the same instant that \( p_a \) did, and they must have appeared on a lap whose boundary contains the critical point they are now on. This fact means that they either appeared in the same lap as bifurcation B or in an adjacent lap for the same \( \lambda \) that caused bifurcation B. This situation is impossible since, as mentioned before, simultaneous bifurcations cannot happen on the same lap (to avoid violating the Schwarzian condition), or on adjacent laps (where the slope prohibits a bifurcation).

Case (b2): Suppose before any fixed point crosses a critical point, some of the laps between the periodic points vanish because critical points fuse. Since all critical points are relative extrema, they must vanish as three consecutive points merging into a single point, i.e., two adjacent laps vanishing. Note that a lap that contains points from bifurcation A or B cannot collapse, because this would cause these points to vanish before they had a chance to fuse. If any number of pairs of intervening laps vanish, we still have an odd number of laps between and so this case reverts to case
(b2). Note that the last lap cannot vanish for it would remove one of the laps with a bifurcation in it as well.

There are no other cases, so it is not possible for two fixed points from unrelated bifurcations of the same period to fuse, and our lemma is proven.

Returning to the proof of Theorem 3.3.4, we note that if the points appear simultaneously, they must cross for each to mature as required, $f^n$ cannot have two (or more) unrelated bifurcations for the same $\lambda$, creating the same period, and we have finished Case 2.

Case 3: We show that we cannot have two simultaneous unrelated bifurcations of differing period, where one is a multiple of the other, at the same value of $\lambda$. This case is can be reduced to Case 2. If one bifurcation gives an orbit of period of length $n$, and the second gives an orbit of period length $kn$. Both of these will be manifested as bifurcations of $f^{kn}$, where the first orbit will be "viewed" as an orbit of length $kn$, i.e. as $k$ repetitions of the length $n$ orbit. Now, we have reduced our problem to Case 2 and this case follows.

Thus, it is impossible for any two non-related bifurcations to happen for any given $\lambda$; our theorem is proved.

This theorem has some immediate consequences that we will list as corollaries.

**Corollary 3.3.4a**: If a periodic point is to spawn a period-doubling bifurcation, it must do so in the very next bifurcation after its creation.

This fact follows because otherwise the new periodic point will have points from an unrelated bifurcation between it and $c$, thus be unable to mature.

**Corollary 3.3.4b**: A periodic point can spawn only one period-doubling bifurcation.
This fact is true because should a point spawn a period-doubling bifurcation twice, by the previous corollary it must do so twice in a row without an intervening bifurcation, this would result in two periodic points with the same itinerary.

The next corollary is concerned with the particular bifurcation that creates the point that will eventually become $c$ for some $\lambda$.

**Corollary 3.3.4c:** Unrelated bifurcations must occur in the following manner near $c$

1) A saddle-node bifurcation must occur between points of the previous bifurcation. In particular, each successive bifurcation must occur between the two points created by the preceding bifurcation in the saddle-node case or between the maturing point and the birthing point in the period-doubling case.

2) A period-doubling bifurcation must form around the maturing point of the previous bifurcation.

The justification of this corollary is found in the fact that the above situations are the only ones that allow each successive bifurcation to mature.

We are now ready to take one last step and connect these bifurcations to the order established by the ordering of their itineraries, starting with a definition.

**Definition 3.3.4:** A periodic orbit's itinerary (versus the itinerary of a periodic point) will be the largest itinerary of its member points.

Note that (as previously proven) the member of a periodic orbit that has the itinerary which represents the entire orbit is the point with the highest numeric value since the ordering of itineraries match the natural ordering of their points numeric value. This member is going to be the point just
after (in the orbit sequence) the point closest to $c$. Hence, the orbit of the point closest to $c$, left-shifted by one must be the highest ranked itinerary of the orbit.

**Lemma 3.3.5a:** The itinerary of the mature orbit is larger than its pre-mature itinerary.

**Proof:** No matter which type of bifurcation occurs, for a small change in the value of $\lambda$ there are two orbits with the same itinerary. Suppose, without loss of generality, the bifurcation occurred to the left of $c$. This means that since the point of the orbit closest to $c$ is to the left of $c$, the itinerary of the orbit must end its period with a 0 entry. In other words, the itinerary has the form $(\overline{A0})$ (the over-bar meaning endless repetition), where $A$ stands for some finite sequence of 1's and 0's. One orbit matures by having one of its points cross $c$, getting the itinerary $(\overline{Ac})$, the kneading sequence. These two itineraries coexist because in the saddle-node case we have the other orbit-pair and in the period-doubling case we have the itinerary of the birthing point. We have by definition of the kneading sequence that $(\overline{A0}) < (\overline{Ac})$, and by the way the ordering was defined we know that $A$ must have an even number of 1's. The mature itinerary will be $(\overline{AI})$ since after crossing $c$ the point is to the right of $c$. By the definition of the ordering, this new itinerary must be larger than the old kneading sequence. So we have $(\overline{A0}) < (\overline{Ac}) < (\overline{AI})$. The case with the bifurcation happening to the right of $c$ the argument is identical. The final sequence of itineraries will be $(\overline{AI}) < (\overline{Ac}) < (\overline{A0})$; with $A$ having an odd number of 1's in this case.

We now come to the final theorem of this section.

**Theorem 3.3.5:** The orbits that mature do so in the order as given by their itineraries.

In other words, if one orbit's itinerary is lower on the ordering than another, it will mature first.
Proof: Since the previously matured orbit, say $M_1$, exists while the next one matures, becoming the kneading sequence for a single $\lambda$, call it $K$. $M_1$ must have an itinerary less than $K$, and the maturing point must (by Lemma 3.3.5a) have a mature itinerary, say $M_2$, larger than $K$. So we have $M_1 < K < M_2$, thus $M_1 < M_2$.

This theorem finishes our exploration of these particular class of functions. Analysis of any particular periodic orbit, what kind of bifurcation created it, and when it forms relative to other periodic orbits can be answered by looking at its itinerary.
4 ~ How Certain Periodic Orbits Imply Others

4.1 Convergent Functions and Existence of Orbits

In a previous section we have shown that through the theorem of Šarkovskii the existence of certain periods imply the existence of other periods. In the preceding section we explored an example of how periodic orbits eventually appear and how, in the case of the example, their appearance relates to the kneading sequence and the ordering of the itineraries. Now, we will look at this area from a slightly different point of view.

Up to this point we have looked at a single parameterized function and shown how orbits appear as the parameter is varied. In this section we will look at two theorems about the properties of functions that are close to each other. All these functions, of course, are members of some transitional family.

Our first theorem and accompanying lemmas and proofs are due to Block and Hart [BH]. It looks at predicting the existence of certain periodic orbits in a function to which a sequence of functions with known periods converges.

**Theorem 4.1.1:** Suppose \( \{f_n\} \) is a sequence of maps in \( C^1(I, I) \) (a super-set of the set of all unimodal functions) converges uniformly to \( f \). If each \( f_n \) has a periodic point \( x_n \) of period \( k \), such that there exists a subsequence of \( x_n \) converges to \( x \), then

1. If \( k \) is odd, \( x \) is a periodic point of \( f \) with period \( k \).
2. If \( k \) is even, \( x \) is a periodic point of \( f \) with period \( k \) or \( \frac{k}{2} \).[BH]

We first need two intermediate lemmas. The first says that if an interval contains all points of a periodic orbit, then it must contain a point whose slope is positive and a point whose slope is less than or equal to negative one.
Lemma 4.1.1a: Let $O(x) = \{x_1, x_2, \ldots, x_k\}$ be the orbit of a point in $I$ of period $k > 2$, such that $x_i < x_{i-1}$. Then there exist points $y$ and $z$ in $[x_1, x_k]$ such that $f'(y) > 0$ and $f'(z) \leq -1$.

Proof: Since the period $k \neq 2$, $x_1$ and $x_k$ cannot both map to each other. So, there must be a point of the orbit between these two points that either maps from one or the other, i.e., there exists an $m$, $1 < m < k$ such that $f(x_m) = x_1$ or $f(x_m) = x_k$. In either case, we can get our necessary point $y$.

If the former is true, then there exists a $y \in [x_m, x_{m+1}]$. In the latter case there exists a $y \in [x_{m-1}, x_m]$ such that $f'(y) > 0$. If $f(x_m) = x_k$, the highest possible value of the orbit, then $f(x_{m-1}) < f(x_m)$. By the Mean Value Theorem, the slope of $f$ is positive for at least some $x$ in the interval. Similarly, if $f(x_m) = x_1$, the lowest possible value of the orbit, then $f(x_m) < f(x_{m+1})$.

Again this situation demands the existence of a positive slope of $f$ in the interval.

Now, we want to find our point $z$. Choose $x_i$, $1 \leq i \leq k$, such that it is the smallest member of the orbit for which $f(x_i) < x_i$ (note $i > 1$). Since $f(x_i) < x_i$, $f(x_i) \leq x_{i-1}$, and by the minimality of $i$ we have $f(x_{i-1}) > x_{i-1}$ which implies $f(x_{i-1}) \geq x_i$. If we have $f(x_i) = x_{i-1}$ and $f(x_{i-1}) = x_i$, the Mean Value Theorem implies that there exists a point, $z \in [x_{i-1}, x_i]$, such that $f'(z) = -1$. If $f(x_i) < x_{i-1}$ or $f(x_{i-1}) > x_i$, we find:

$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} < \frac{x_i - x_{i-1}}{x_i - x_{i-1}} = 1.$$

Again using the Mean Value Theorem, there exists a point, $z \in [x_{i-1}, x_i]$, such that $f'(z) < -1$.

Combining the cases, we have $z$ with the property that $f'(z) \leq -1$, and our lemma is proved.
Using, the above lemma, we next prove a lemma that gives us one last crucial piece necessary for the proof of our theorem.

**Lemma 4.1.1b:** Suppose \( \{f_n\} \) is a sequence of maps in \( C^1(I,I) \) that converges uniformly to \( f \). If each \( f_n \) has a periodic point, \( x_n \), of period \( k > 2 \) such that \( x_n \) converges to \( x \), then \( x \) is a fixed point of \( f^k \) but not of \( f \).

**Proof:** Notice that the \( x_n \)'s are fixed in their respective \( f_n^k \). If \( f_n \) converges to \( f \) uniformly, then \( f_n^k \) must converge to \( f^k \) uniformly. We will show this fact by an induction argument on \( k \). Clearly it is true for the case \( k = 1 \), since \( f_n \) converge to \( f \) uniformly. We now assume that \( f_n^{k-1} \) converges to \( f^{k-1} \) uniformly and then prove that \( f_n^k \) converges to \( f^k \) uniformly. So, we want to show that for all \( \epsilon > 0 \), there exists an \( N \), for all \( x \), such that for all \( n \geq N \), \( |f_n^k(x) - f^k(x)| < \epsilon \).

Let \( \epsilon > 0 \) be given. By the continuity of \( f_n \), we know that there exists \( \delta_1 > 0 \) such that \( |x_1 - x_2| < \delta_1 \Rightarrow |f_n(x_1) - f_n(x_2)| < \epsilon/2 \). Our induction hypothesis now gives us that there exists \( N_1 \) such that for all \( n \geq N_1 \), \( |f_n^{k-1}(x) - f^{k-1}(x)| < \delta_1 \) for all values of \( x \). Thus, by putting these two statements together, we have that for all \( n \geq N_1 \), \( |f_n(f_n^{k-1}(x)) - f_n(f^{k-1}(x))| < \epsilon/2 \) for all values of \( x \). From our given convergence of \( f_n \) to \( f \) we get that

\[
\exists N_2 \ni \forall n \geq N_2, \quad |f_n(f^{k-1}(x)) - f(f^{k-1}(x))| < \epsilon/2.
\]

Let \( N = \max(N_1, N_2) \). We get that

\[
\forall n \geq N, \quad |f_n(f_n^{k-1}(x)) - f_n(f^{k-1}(x))| + |f_n(f^{k-1}(x)) - f(f^{k-1}(x))| < \epsilon,
\]

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and the triangle inequality gives us for all \( n \geq N, \quad |f_n(f_n^{-1}(x)) - f(f_n^{-1}(x))| < \varepsilon \), which is the same as: for all \( n \geq N, \quad |f_n^k(x) - f^k(x)| < \varepsilon \) for all values of \( x \), i.e., \( f_n^k \) converges to \( f^k \) uniformly.

Note that \( x \) is fixed in \( f_n^k \), so \( x \) must be fixed in \( f^k \). We know that for all \( n \geq N, \quad |f_n^k(x) - f^k(x)| < \varepsilon \). Since \( f_n^k(x) = x \) for all \( n \), we can substitute this into our inequality and get that for all \( n \geq N, \quad |x - f^k(x)| < \varepsilon \). Both \( x \) and \( f^k(x) \) are constants whose difference is arbitrarily small, so their difference must be zero. Hence \( f^k(x) = x \), and \( x \) is a fixed point of \( f^k \).

Next we will assume that \( x \) is fixed in \( f \) and arrive at a contradiction.

Let \( p_n = \min O(x_n) \) and \( q_n = \max O(x_n) \). Since we have an infinite collection of functions with finite periods, we have \( f_n^i(x_n) = p_n \) and \( f_n^j(x_n) = q_n \) for an infinite number of \( n \)'s (\( i, j \in \mathbb{Z}^+ \)).

For the rest of this proof, we will look at the subsequence of those \( f_n \)'s for which the previous fact holds. Since \( f_n \) converges to \( f \) and \( x_n \) converges to \( x \), we can construct a proof similar to the previous argument and get \( f_n^i(x_n) \) converging to \( f^i(x) = p \) and get \( f_n^j(x_n) \) converging to \( f^j(x) = q \).

By our assumption of \( x \) being fixed by \( f \), we have \( p = q = x \). So, both \( p_n \) and \( q_n \) converge to \( x \).

All the \( f_n \)'s \( x_n \) have period greater than 2, hence we can apply Lemma 4.1.1a to each of these functions. We find that there exists two points \( y_n, z_n \in [p_n, q_n] \) such that \( f_n^i(y_n) > 0 \) and \( f_n^j(z_n) \leq -1 \) for infinite \( n \)'s; the different slopes imply that \( y_n \neq z_n \). Suppose, without loss of generality, that \( y_n < z_n \) for all \( n \), then \( p_n \leq y_n < z_n \leq q_n \). Since the \( f_n \)'s are continuous, we have a sandwiching effect forcing both \( y_n \) and \( z_n \) to converge to \( x \). Noting that all involved functions are continuous and converge to \( f \), we have that \( f'(x) \geq 0 \) and \( f'(x) \leq -1 \), which is a contradiction. Thus \( x \) cannot be fixed in \( f \).

\[\square\]

Now we are ready to prove the theorem.
Proof of Theorem 4.1.1: We have three cases: where \( k \) is odd, where \( k \) is a power of 2, and \( k \) is the product of an odd number and a power of 2.

Case 1: \( k \) is odd. Our proof will follow an induction argument; we will prove for \( k = 3 \), assume for every case less than \( k \) and prove our theorem true for \( k \). If \( k = 3 \), we apply Lemma 4.1.1b and are done since if \( x \) is not fixed under \( f \) but is fixed under \( f^3 \), it must be a point of period 3.

Now, we assume the theorem holds for all odd numbers less than \( k \) and we will show that it hold for \( k \). So, for all odd numbers less than \( k \), assume that \( x \) is a periodic point of \( f \) with period \( k \).

By Lemma 4.1.1b we know that \( x \) is fixed under \( f^k \) but not under \( f \), thus \( x \) must be periodic under \( f \) with a period of a length that is a factor of \( k \) (not 1), say \( r \). So let \( k = r \cdot s \). Since \( k \) is odd, both \( r \) and \( s \) must be odd, and \( s \) must be less than \( k \) since \( r \neq 1 \).

Notice that \( x_n \) has a period of \( s \) under \( f_n^r \) (just as an orbit of period 6 under some function \( g \) is periodic with period 2 under \( g^3 \) and periodic of period 3 under \( g^2 \)). As previously shown, \( f_n^r \) converges to \( f^r \). Since \( s \) is an odd period less than \( k \), it falls under our induction hypothesis from which we can conclude that \( x \) has period \( s \) under \( f^r \). Recall that \( x \) is fixed under \( f^r \), because it has a period of length \( r \) under \( f \). These two facts force \( s = 1 \), which makes \( r = k \). Thus the point \( x \) has a period of \( k \).

Case 2: \( k \) has a period of the form \( 2^s \), \( s \geq 0 \). If \( k = 1 \), we note that part of the proof of Lemma 4.1.1b showed us that if the \( x_n \)'s are fixed then \( x \) is fixed under \( f \). If \( k = 2 \), we apply reasoning from Lemma 4.1.1b, we see that \( x \) is fixed under \( f^2 \) but not under \( f \), thus \( x \) must have a period of two under \( f \).

Suppose \( k \geq 4 \). Recalling that \( k = 2^s \), we make the following construction: \( g_n = f_n^{k/4} \) and \( g = f^{k/4} \). Observe that \( x_n \) has a period length 4 under \( g_n \). By Lemma 4.1.1b, \( x \) is fixed under \( g^4 \).
but not under $g$, hence $x$ must be periodic under $g$ with a period of 2 or 4. If $x$ is has a period of 2 under $g$, it has a period of $k/2$ under $f$. If $x$ has a period of 4 under $g$, it has a period of $k$ under $f$.

Case 3: $k$ has a period in the form of $m \cdot 2^s$, $s \geq 1$ and $m$ is odd and greater than three. Consider the following construction: let $r = 2^s$, $g_n = f^r_n$ and $g = f^r$. Now, $x_n$ has odd period $m$ under $g_n$, thus has period $m$ under $g$ by Case 1. So, $x$ has period $m$ under $f^r$. From a different viewpoint consider this construction: let $h_n = f^{m^n}$ and $h = f^m$. This time $x_n$ has a period of $r$ under $h_n$ thus $x$ must have period of $r$ or $r/2$ under $h$ (thus $f^m$) by Case 2.

Let the period of $x$ under $f$ be length $t$. First, note that $r$ and $m$ are relatively prime and so are $m$ and $r/2$. Since $x$ has period $m$ under $f^r$, $t$ must be a multiple of $m$. Similarly since $x$ has have period of $r$ or $r/2$ under $f^m$, $t$ must be a multiple of $r$ and $r/2$. Recall that $x$ is fixed under $f^k$, so $t$ must be a factor of $k$. Finally, recall that $k = m \cdot r$. Putting these last facts together, we get that $t = m \cdot r = k$, or $t = m \cdot r/2 = k/2$, and Case 3 is done.

Thus the theorem is proved.

4.2 Perturbations and the Existence of Orbits

The theorem of this section considers the existence of periods in functions that are close to a function with known periods. This theorem, its associated lemmas, and all the proofs are due to Block.[LB]

The first preliminary lemma infers the existence of a period 3 orbit from the existence of a special kind of period 4 orbit and will use Markov graphs.

**Lemma 4.2.1a:** Suppose $f \in C^0(I, I)$ has a point $p$ of period 4 such that its orbit, $O(p) = \{p_1, p_2, p_3, p_4\}$, with the following properties: $p_1 < p_2 < p_3 < p_4$ and $f(\{p_1, p_2\}) \neq \{p_3, p_4\}$. Then $f$ has a point of period 3.
Proof: We will prove this by exhaustion of cases. First, there are \(4!/4 = 6\) possible ways for these four points to map to one another. Of these, two are excluded by our hypothesis (namely, (1) when the first point maps to the third and the second maps to the fourth and (2) vice versa). So, we only have four cases: \(p_1 \to p_3 \to p_4 \to p_2\), \(p_1 \to p_4 \to p_3 \to p_2\), \(p_1 \to p_2 \to p_3 \to p_4\), and \(p_1 \to p_2 \to p_4 \to p_3\). In each case we will appeal to the continuity of the function to get the mapping of each of the three intervening intervals: \(I_i = [p_i, p_{i+1}]\). From these we will construct a fixed point of \(f^3\) that cannot be fixed under \(f\) and thus must be a point of period 3. Note that such a point cannot be an endpoint of any \(I_i\), because that is on a period 4 orbit.

Case 1: \(p_1 \to p_3 \to p_4 \to p_2\). In this case we have \(f(I_1) \supset I_1 \cup I_2, f(I_2) \supset I_1 \cup I_2 \cup I_3,\) and \(f(I_3) \supset I_2 \cup I_3\). Consider the following path through the resulting Markov graph: \(I_1 \to I_1 \to I_2 \to I_1\). There exists \(J \subset I_1\) such that \(f^2(J) \subset I_2\) and \(f^3(J) \supset I_1 \supset J\), which implies a fixed point of \(f^3\) that leaves the interval \(I_1\) during its orbit, thus it cannot be fixed under \(f\) and must be a point of period 3.

Case 2: \(p_1 \to p_4 \to p_3 \to p_2\). In this case we have \(f(I_1) \supset I_1 \cup I_2 \cup I_3, f(I_2) \supset I_1,\) and \(f(I_3) \supset I_2\). Consider the path of a point through \(I_1 \to I_1 \to I_2 \to I_1\). By the same reasoning in case one \(f\) has a point of period 3.

Case 3: \(p_1 \to p_2 \to p_3 \to p_4\). In this case we have \(f(I_1) \supset I_2 \cup I_3, f(I_2) \supset I_1 \cup I_2 \cup I_3,\) and \(f(I_3) \supset I_1 \cup I_2\). Consider the path of a point through \(I_1 \to I_2 \to I_3 \to I_1\). We have \(f^3(I_1) \supset I_1\), and again by the above reasoning we have that \(f\) has a point of period 3.

Case 4: \(p_1 \to p_2 \to p_4 \to p_3\). In this case we have \(f(I_1) \supset I_2, f(I_2) \supset I_3,\) and \(f(I_3) \supset I_1 \cup I_2 \cup I_3\). Again, consider the path of a point through \(I_1 \to I_2 \to I_3 \to I_1\), and \(f\) has a point of period 3 and our lemma proved.
The theorem also appeals to the following property, which says that if an interval maps over the top of another interval by way of a continuous function, the continuity of that function implies that a subinterval of the first interval maps onto the second.

**Property 4.2.1:** Let \( f \) be continuous and \( J \) and \( K \) be closed intervals such that \( f(J) \supseteq K \), then there exists a closed interval, \( L \subseteq J \) such that \( f(L) = K \).

For this theorem we introduce a notation used by Block [LB] to denote the set of periods of a given functions.

**Definition 4.2.1:** \( P(f) \) is the set of integers that correspond to the lengths of all periods that \( f \) has.

**Theorem 4.2.1:** Let \( f \) belong to \( C^0(I, I) \) and suppose that \( n \in P(f) \). There exists a neighborhood \( N \) of \( f \) in \( C^0(I, I) \) (under the uniform topology) such that every function in \( N \) has all periods that are to the right of \( n \) in the Šarkovskii ordering.

This proof consists of a series of five lemmas, the main result being broken down into two cases that are proved by Lemmas 4.2.1d and 4.2.1f. The first lemma give us a powerful tool in understanding perturbations and the existence of orbits. By previous arguments, we know that certain perturbations can cause orbits to come into and go out of existence. Within such perturbations, there are no neighborhoods of the function that will leave all the function’s orbits unchanged. The following lemma creates a structure that can survive small perturbations.

**Lemma 4.2.1b:** Let \( f \in C^0(I, I) \). Let \( k \geq 3 \) be an odd positive integer. Suppose there exists a point \( y \) in \( I \) such that:

1) \( f^{k-2}(y) < f^{k-4}(y) < f^{k-6}(y) < \ldots < f^3(y) < f(y) < y \).

2) \( y < f^2(y) < f^4(y) < f^6(y) < \ldots < f^{k-1}(y) \).
3) \( y < f^k(y) \).

Then \( f \) has a period of length \( k \), i.e. \( k \in P(f) \).

To clarify we note that part 1 states that every odd step, the orbit of \( y \) is getting closer to zero (\( y \) confined to \( I \) and the orbit confined to \( k \) steps). Part 2 says that on every even step, the orbit of \( y \) is getting closer to one. These two parts along with part 3 says that the orbit of \( y \) is oscillating outward from \( y \), with the \( k \)th step ending up larger than \( y \).

**Proof:** From the hypothesis we note that \( f(y) < y \) while \( f(f(y)) = f^2(y) > y \). These facts along with the continuity of \( f \) give us that \( f([f(y),y]) \supset [f(y),y] \). Hence \( f \) has a fixed point \( e \in (f(y),y) \). (The fixed point cannot be at the endpoints, since \( y \)'s orbit is clearly not fixed.) Let \( M_1 = [e,y] \), \( M_3 = [y,f^2(y)] \), \ldots, and \( M_k = [f^{k-3}(y),f^{k-1}(y)] \), i.e., \( M \)'s with all odd subscripts.

Construct the \( M \)'s with even-subscripts by: \( M_2 = [f(y),e] \), \( M_4 = [f^3(y),f(y)] \), \ldots, and \( M_{k-1} = [f^{k-2}(y),f^{k-4}(y)] \).

Notice that \( f(M_i) = M_{i+1} \) for \( 1 \leq j \leq k - 1 \) because \( f \) maps each endpoint of \( M_i \) to different endpoints of \( M_{i+1} \). Also, \( f(M_k) \supset M_1 \), since \( f(f^{k-3}(y)) = f^{k-2}(y) < f(y) < e \) and \( f(f^{k-1}(y)) = f^k(y) > y \). Thus by Property 4.2.1 we have a closed interval \( K_k \subset M_k \) such that \( f(K_k) = M_1 \). Now, \( f(M_{k-1}) = M_k \supset K_k \), so by Property 4.2.1 we have a closed interval \( K_{k-1} \subset M_{k-1} \) such that \( f(K_{k-1}) = K_k \) and \( f^2(K_{k-1}) = M_1 \). Continuing this construction back to \( M_1 \) we form an interval, \( K_1 \), such that \( f^k(K_1) = M_1 \). So, there exists a point \( z \) in \( K_1 \) fixed under \( f^k \).

So, \( z \) must have a period of length \( k \), a factor of \( k \), or 1. Suppose \( z \) has a period of \( 1 \leq j < k \); then \( f^j(z) = z \). Since \( z \in K_1 \) and \( f^j(K_1) = K_{j+1} \subset M_{j+1} \), we have \( f^j(z) \in M_{j+1} \) and \( z \in M_{j+1} \). The only way this situation can occur is if \( j \) were 1 or 2, since \( M_1 \) shares points only with \( M_2 \) and \( M_3 \). If \( j = 1 \) then \( z = e \), the only point common between \( M_1 \) and \( M_2 \). This case is not possible since
e is not in $M_3$ and $z$ is. In the other case, when $j = 2$, we get $z = y$, the only common point between $M_1$ and $M_3$. This case is also not possible since, clearly $y$ is a point of period at least $k$ (if it is a periodic point at all). Therefore, $z$ must have a period of $k$, the only remaining possibility, and our lemma is proved.

Notice that it is possible to reverse the direction of all the inequalities and the argument, with slightly different construction of the $M'$s, would give the same result.

The important implication of this lemma is that the structure created as the hypothesis can withstand perturbations. All we have is a series of relationships between one member of an orbit and another. If $x < f(x)$, this means that at the point $x$, $f$ lies above the line $y = x$. If the inequality is reversed, the function lies below the line $y = x$ at the point $x$. In either case the function can perturb a bit and still remain on the same side of $y = x$.

Next, we prove the theorem by looking at two cases. These will be Lemmas 4.2.1d and 4.2.1e. The cases are when $f$ has a period that is an odd multiple of a power of 2, and when it has a period that is a power of 2 only. Both cases are proved in two steps. The first case is when $f$ has a period of an odd multiple of a power of 2; the power allowed to be zero, hence allowing for the case when the period is odd. The preliminary lemma looks at the what can be said when $f$ has an odd period.

**Lemma 4.2.1c:** Let $f \in C^0(I, I)$. If $f$ has a point of odd period, $n$, with $n \geq 3$, then there exists a neighborhood $N$ around $f$ in $C^0(I, I)$ such that if $g \in N$, then $(n + 2) \in P(g)$, i.e., $g$ has a point of period $n + 2$.

**Proof:** This proof will use Theorem 3.2.1 (Štefan’s Theorem) to get the hypothesis of the previous lemma. Note that if we look at the minimal period in the Šarkovskii ordering of $f$, say $m$,
it must be an odd number less than \( n \). Should \( g \) have period \( m + 2 \), which is either \( n \) or to the left of it in Šarkovskii’s ordering, Šarkovskii’s Theorem (Theorem 3.1.1) tells us that \( g \) must have a period of everything to the right of \( m + 2 \) in Šarkovskii’s ordering, namely \( n + 2 \). Thus we need only prove the case for \( n \) being the minimal (leftmost) period under the Šarkovskii ordering.

Let the orbit of period \( n \) be given by \( \{p_1, p_2, \ldots, p_n\} \) with \( p_1 < p_2 < \ldots < p_n \). Then due to the minimality of \( n \), Štefan’s Theorem applies. This theorem gives, as its result, two cases: Case A or Case B. By the note in the previous lemma, for our purposes these cases both satisfy the hypothesis for lemma 4.2.1b. We will take Case A (Case B is proved similarly) from Štefan’s Theorem: let \( t = (n + 1)/2 \), then \( f(p_{t-i}) = p_{t+i} \) for \( 1 \leq i \leq t - 1 \), \( f(p_{t+i}) = p_{t+i-1} \) for \( 0 \leq i \leq t - 2 \), and \( f(p_n) = p_t \). If we let \( z = p_t \), then we can restate the results from Štefan’s Theorem in the following form:

1) \( f^{n-2}(z) < f^{n-4}(z) < f^{n-6}(z) < \ldots < f^3(z) < f(z) < z \).

2) \( z < f^2(z) < f^4(z) < f^6(z) < \ldots < f^{n-1}(z) \).

3) \( z = f^n(z) \).

Notice that this is almost what we need for the hypotheses of lemma 4.2.1b, we just need a ‘less than’ relation in part 3 instead of the equality.

Since \( f^2(z) > z \) and \( f(z) < z \), we see that \( f([f(z), z]) \supset [f(z), z] \). So, there exists a point, \( b \), such that \( b \in (f(z), z) \) and \( f(b) = z \). Since \( f(z) < b < z = f(b) \), we have \( f([b, z]) \supset [b, z] \). So, there is a point, \( y \), such that \( y \in (b, z) \) and \( f(y) = b \), which means that \( f^2(y) = z \). Note that \( y < z \), hence \( y < f^n(z) = f^{n-2}(y) \). Now we rewrite the results from Štefan’s Theorem replacing \( z \) with \( f^2(y) \) as well as noting that \( f^2(y) > y > f(y) \) and we get:

1) \( f^{n+2-2}(y) < f^{n+2-4}(y) < f^{n+2-6}(y) < \ldots < f^3(y) < f(y) < y \).
2) \( y < f^2(y) < f^4(y) < f^6(y) < ... < f^{n-2}(y) \).

3) \( y < f^{n-2}(y) \).

We now have constructed a point, \( y \), that satisfies the hypotheses of Lemma 4.2.1b with \( k = n + 2 \). So, by that lemma, \( f \) has a point of period \( n + 2 \).

As noted previously, the above structure can withstand small perturbations. So, there is some neighborhood, \( N \), of \( f \) (in the uniform topology), where the above structure still holds for some point, \( z \), for each \( g \) in \( N \). Thus all such \( z \) satisfies the hypotheses of Lemma 4.2.1b with \( k = n + 2 \).

So by Lemma 4.2.1b, for all \( g \) in \( N \), \( n + 2 \in P(g) \), proving the lemma.

Next we look at the second step in proving our first case by this next lemma.

**Lemma 4.2.1d:** Let \( f \in C^0(I,I) \). If \( f \) has a point of period \( n = r \cdot s \), then there exists a neighborhood, \( N \), around \( f \) in \( C^0(I,I) \) such that if \( g \in N \), then \( g \) has a periodic point of every length to the right of \( n \) in the Šarkovskii ordering.

**Proof:** First, since \( f \) has a point of period \( n = r \cdot s \), \( f^r \) must have a point of period \( s \). Since \( s \) is odd and greater than 1, it satisfies the hypothesis of the previous lemma. So, there exists a neighborhood of \( f^r \), say \( N_1 \), with the property that if \( h \) is in \( N_1 \), it must have a point with the period of \( s + 2 \).

Since all our functions (and thus their compositions) are in \( C^0(I,I) \), they must map at least into \( I \). Now, \( g \) is in a neighborhood of \( f \), say \( N \) with radius \( \epsilon_1 \), if \( \max_{x \in I} |g(x) - f(x)| < \epsilon_1 \). Notice that \( \max_{x \in I} |g(g^{-1}(x)) - f(f^{-1}(x))| \leq \max_{x \in I} |g(x) - f(x)| < \epsilon_1 \). So, there is a neighborhood of \( f \), say \( N \), such that if \( g \) is in \( N \), then \( g^r \) is in \( N_1 \). Let this be the case, then \( g^r \) must have a point of period \( s + 2 \). So, \( g \) must have a point of period \((s + 2) \cdot j\), where \( j \) is a factor of \( r \), i.e. \( j = 2^t \), \( 0 \leq t \leq i \).

Now, \((s + 2) \cdot j\) is either \( n \), to the left of \( n \) on the Šarkovskii ordering, or it is immediately to the right...
of $n$. Šarkovskii's Theorem states that the existence of an orbit length implies the existence of orbits of all lengths to the right in the ordering. So, the worst case is the existence of all orbit lengths starting immediately to the right of $n$ and to the right in the Šarkovskii ordering, thus our lemma is proved.

The lemma has proved our theorem for the case when $n$ has the form $n = r \cdot s$, where $r = 2^i$, $i$ being an nonnegative integer and $s$ being an odd integer larger than one. Now, we consider the second case: $n$ is a period of length equal to a power 2.

**Lemma 4.2.1e:** Let $f \in C^0(I,I)$. Let $f$ have a point of period 4, then there exists a neighborhood, $N$, around $f$ in $C^0(I,I)$ such that if $g \in N$, then $g$ has a point of period 2.

**Proof:** Label the period-four orbit: $\{p_1, p_2, p_3, p_4\}$ with $p_1 < p_2 < p_3 < p_4$. Should $f$ have a point of period 3, then by Lemma 4.2.1d, we get a neighborhood of $f$ in which all members have all periods but 3, hence they have a period of 2.

Suppose $f$ has no point of period 3. Applying the contrapositive of Lemma 4.2.1a, we get that $f(\{p_1, p_2\}) = \{p_3, p_4\}$, which implies that $f(\{p_3, p_4\}) = \{p_1, p_2\}$. So, $f(\{p_1, p_2\}) \supseteq \{p_3, p_4\}$ and $f(\{p_3, p_4\}) \supseteq \{p_1, p_2\}$. Thus there exists $K \subseteq \{p_1, p_2\}$ such that $f(K) = \{p_1, p_2\}$. Let $v \in \{p_1, p_2\}$ such that $f(v) = p_4$. Obviously $x \neq p_4 = f(v) = f^2(x)$, since should $x$ equal $p_4$, then $x$ would have a period of 2, but it lies on an orbit of period 4. So, $x < p_4$, which means that $f^2(x) > x$. Similarly let $u \in \{p_1, p_2\}$ such that $f(u) = p_3$. Again, since $f(K) = \{p_1, p_2\}$, there is a $y$ in $K$ such that $f(y) = u$. Now, $y \neq p_3 = f(u) = f^2(y)$, since should $y$ equal $p_3$, we have the identical problem of $y$ having period 2 when it lies on an orbit of period 4. So, $y > p_3$, thus $f^2(y) > y$. Notice that $f(z) < p_3$ for all $z$ in $K$. 68
Now, \( f^2([x,y]) \supseteq [x,y], \) which implies the existence of a point in that interval that is either fixed or on an orbit of period 2, under \( f. \) Since \( f([x,y]) \subseteq [p_1,p_2], \) the point cannot be fixed under \( f, \) so it must be of period 2.

For the same reasons as mentioned in the comment on Lemma 4.2.1b, the inequalities can withstand some perturbation. Thus, there exists a neighborhood of \( f, \) say \( N, \) where any \( g \) in \( N \) we have the same relationship: \( g^2(x) > x, \) \( g^2(y) > y, \) and \( g(z) < p_3 \) for all \( z \) in \( K. \) So, for all \( g \) in \( N, \) \( g \) has a point of period 2, and the lemma is done.

**Lemma 4.2.1f:** Let \( f \in C^0(I,I). \) If \( f \) has a point of period \( n = 2^m, \) \( m \) a positive integer, then there exists a neighborhood \( N \) of \( f \) in \( C^0(I,I) \) such that if \( g \in N, \) then \( g \) has a periodic point of every length to the right of \( n \) in the Šarkovskii ordering.

**Proof:** If \( m \) is one, then \( n \) is 2, and all we need to show is that our neighborhood contains functions with a fixed point. This result is immediate. Let the orbit of period 2 be \( \{p_1,p_2\}, \) and define \( J = [p_1,p_2]. \) Clearly \( f(J) = J, \) so there is a fixed point of \( f \) in \( J; \) call it \( p. \) Now, there is a point \( x \) in \( J \) such that \( x > p \) and \( f(x) < p \) thus \( x > f(x). \) Similarly, there is a point \( y \) in \( J \) such that \( y < p \) and \( f(y) > p \) thus \( y < f(y). \) As before, there is a neighborhood \( N \) of \( f \) such that for all \( g \) in \( N \) the same relationships hold: \( x > g(x) \) and \( y < g(y). \) These relationships reveal that \( g([x,y]) \supseteq [x,y], \) which implies the existence of a fixed point of \( g \) and this case is proved.

Now assume that \( m > 1. \) If we let \( r = 2^m - 2, \) then \( m = 4r \) and \( f^r \) has a point of period 4. We apply Lemma 4.2.1e and note that there exists a neighborhood, \( N_1, \) around \( f^r \) such that all functions within \( N_1 \) have a point of period 2. Also by the same argument found in Lemma 4.2.1d, there is a neighborhood, \( N, \) around \( f \) such that if \( g \) is in \( f, \) \( g^r \) is in \( N_1. \) Let \( g \) be in \( N, \) then since \( g^r \) has a point of period 2, \( g \) must have a point of period \( 2 \cdot 2^m - 2, \) or \( 2^m - 1. \) This period is immediately to the right.
of \(2^m\) in the Šarkovskii ordering, and Šarkovskii’s Theorem gives us all the periods further right in the ordering. This fact proves our theorem for the case when \(n\) is a power of 2, which is the final case, thus our lemma is proved.

And the lemma finishes the proof of the theorem.
In this section we will look at the three parts of Devaney's definition of Chaos. In general we can eliminate one of the conditions as being implied by the other two, and in special cases we can eliminate all but topological transitivity.

### 5.1 Sensitive Dependence on Initial Conditions

After Devaney proposed his definition of chaos, Banks, et. al., [JB] proved that topological transitivity and existence of dense periodic orbits implies the sensitive dependence on initial conditions. The following theorem and proof are due to Banks, et. al. [JB]:

**Theorem 5.1.1:** If $f: I \to I$, continuous, is transitive and has dense periodic points, then $f$ has sensitive dependence on initial conditions.

**Proof:** The original theorem sits in a more general setting than we need and has been modified to suit our needs, but observe that $I$ need only be a metric space for this proof to work. We assume topological transitivity and density of periodic orbits and will show sensitive dependence. The definition of sensitive dependence on initial conditions requires a sensitivity constant $\delta$, so our first task is to come up with one.

Notice that since the periodic points are dense on the interval, given any point on the interval there exists a periodic point that is at least one-half an interval away from this given point. In fact, take any two different periodic points $q_1$ and $q_2$, which lie on different orbits, then let $\delta_0 = \min\{|x - y|: x \in O(q_1), y \in O(q_2)\}$. Notice that any point in $I$ is at least $\delta_0/2$ away from any point in one of the two orbits. Let the sensitivity constant be $\delta = \delta_0/8$. We will use the standard notation of $B_\delta(x)$ to denote an open ball (set) around $x$ of radius $\delta$. 

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Let $x$ be any point in $I$ and let $N$ be some neighborhood around $x$. Let $U = N \cap B_\delta(x)$. By
the density of periodic points, there is a periodic point $p$ in $U$. Let $n$ be the period of $p$. Next we note
that either the orbit of $q_1$ or $q_2$ is at least $\delta_0/2$ or $4\delta$ away from $x$. Which ever one it is, let $q$ be a
member of its orbit. So all points in $O(q)$ are at least $4\delta$ away from $x$. Let $V = \bigcap_{i=0}^{n} f^{-i}(B_\delta(f^i(q)))$.
Notice that $V$ is a neighborhood of $q$ such that for $i \leq n$, $f^i(V) \subset B_\delta(f^i(q))$.

By the topological transitivity of $f$, there is a point $y$ in $U$ and an integer $k$ such that
$f^k(y) \in V$. Let $j$ be the integer part of $(k/n) + 1$. Since $j \leq (k/n) + 1$, we have $nj - k \leq n$ and
$nj - k \geq 1$.

Consider the following construction: $f^{nj}(y) = f^{nj-k+k}(y) = f^{nj-k}(f^k(y))$. Since $f^k(y) \in V$,
$f^{nj-k}(f^k(y)) \in f^{nj-k}(V)$. Also since $nj - k \leq n$, we have by the construction of $V$
$f^{nj-k}(V) \subset B_\delta(f^{nj-k}(q))$, which implies $f^{nj}(y) \in B_\delta(f^{nj-k}(q))$, since $f^{nj}(y) \in f^{nj-k}(V)$. Recall
that $p$ has a period if $n$, so $f^{nj}(p) = p$.

Thus $|p - f^{nj}(y)| = |f^{nj}(p) - f^{nj}(y)|$. We use the triangle inequality in its subtraction form
and get $|p - f^{nj}(y)| \geq |f^{nj}(y) - x| - |x - p|$. Applying the triangle inequality a second time on
the first element of the difference, we get $|f^{nj}(y) - x| \geq |x - f^{nj-k}(q)| - |f^{nj-k}(q) - f^{nj}(y)|$.
Putting these two pieces together, we get:

$$|f^{nj}(p) - f^{nj}(y)| \geq |x - f^{nj-k}(q)| - |f^{nj-k}(q) - f^{nj}(y)| - |p - x|.$$ 

Now, $p \in B_\delta(x)$ so $|p - x| < \delta$. Similarly $f^{nj}(y) \in B_\delta(f^{nj-k}(q))$ implies $|f^{nj}(y) - f^{nj-k}(q)| < \delta$.
And as previously shown $|x - f^{nj-k}(q)| > 4\delta$. Thus $|f^{nj}(p) - f^{nj}(y)| > 4\delta - \delta - \delta = 2\delta$.

Finally we use the triangle inequality once more to get:
\[ |f^n(p) - f^n(x)| + |f^n(x) - f^n(y)| \geq |f^n(p) - f^n(y)| > 2\delta. \]

This fact implies that either \( |f^n(p) - f^n(x)| > \delta \) or \( |f^n(x) - f^n(y)| > \delta \). Both \( y \) and \( p \) are in \( N \) and one of them in \( n \) iterations is more than \( \delta \) away from \( x \), in other words, \( f \) has sensitive dependence on initial conditions, and our proof is done. \( \square \)

5.2 Transitivity and the Dense Orbit

Now, we will look at a theorem that shows that topological transitivity is equivalent to the existence of a dense orbit and then use this information in the final theorem of this section to show that in certain cases topological transitivity is sufficient to imply chaos. The following equivalence was stated by Devaney [RD]. The strategy of the proof was mentioned by Devaney and we also use strategies from a similar theorem by Barge and Martin [BM1].

**Theorem 5.2.1:** Let \( f \) be unimodal, then \( f \) is topologically transitive on \( I \) if and only if \( f \) has an orbit dense on \( I \).

**Proof:** We need to prove two implications and will start with the forward implication. Let \( f \) have an orbit, \( O(x) \), dense in \( I \) and then show that \( f \) is topologically transitive. Given the open intervals \( U \) and \( V \), subsets of \( I \), we need to show that there exists an \( N \) such that \( f^N(U) \cap V \neq \emptyset \).

Since \( O(x) \) is dense, there exists a point, \( y \), in \( U \) such that \( y \in O(x) \). Since \( y \) is a member of a dense orbit there exists an \( N \) such that \( f^N(y) \in V \). Thus, \( f^N(U) \cap V \neq \emptyset \), and we are done.

The reverse implication is not so straightforward. The essence of this part of the proof is to assume there is no dense orbit, then use two lemmas to eliminate all but periodic and eventually periodic orbits from \( I \). Next, we will establish that there is only a countable number of remaining
orbits, all consisting of a finite number of points. Finally, we will get that $I$, a sub-interval of the real line, has a countable number of members (or a measure of zero), which contradicts the un-countability of the real numbers.

Let $f$ be topologically transitive on $I$. Assume that there is no dense orbit on $I$.

**Lemma 5.2.1a:** Let $f$ be topologically transitive, then if an orbit is dense on any sub-interval of $I$ it must be dense on the entire interval.

**Proof:** Let $J$ be an open sub-interval of $I$, and let $O(x)$ be an orbit dense in $J$. Let $V$ be any open sub-interval of $I$. If we can show that a member of $O(x)$ is in $V$, we will have shown that the orbit is dense on $I$. Since $f$ is topologically transitive, there is an $N$ such that $f^N(J) \cap V \neq \emptyset$. This intersection is an open interval, since it is formed by two open intervals. So, there exists an open sub-interval, $U$, of $J$ such that $f^N(U) \subset V$. Since $O(x)$ is dense in $U \subset J$, there must be a point, $y$, in $U$ that is a member of the dense orbit. Now, notice that $f^N(y) \in V$. Hence, $V$ has a member of $O(x)$; thus the orbit is dense on $I$. □

**Lemma 5.2.1b:** Let $f$ be unimodal. If $f$ is topologically transitive, then any orbit that is asymptotic, has as its limit a member of a dense orbit.

**Proof:** Let the hypothesis hold and let $O(x)$ be an asymptotic orbit. So, some sub-sequence of $O(x)$ converges to a point, call it $y$.

Assume that $y$ does not lie on an asymptotic orbit itself, since if it did, its limit would be shared by the first orbit, i.e., if $O(y) \to O(z)$, and $O(z) \to w$, then $O(y) \to w$ (or some sub-sequence of these orbits). By $O(y) \to O(z)$, we are saying that a subsequence of the first orbit, say $\{y_1, y_2, \ldots\}$, and a subsequence of the second orbit, say $\{z_1, z_2, \ldots\}$ have the following property: there exists an $N_1$ such that for all $n > N_1$, $|y_n - z_n| < \varepsilon/2$. 

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If $O(z) \to w$ then there exists an $N_2$ such that for all $n > N_2$, $|z_n - w| < \epsilon/2$. If we let $N = \max\{N_1, N_2\}$ and use the triangle inequality, we get: $|y_n - w| < |y_n - z_n| = |z_n - w| < \epsilon$, for all $n > N$, i.e., $O(y) \to w$.

So we can get a limit point that is itself not on an asymptotic orbit. If $y$ is not on an asymptotic orbit, it must be on one of the following types of orbits: periodic, eventually periodic, or wandering—hence a dense orbit.

Assume that $O(y)$ is not wandering. Then it has a finite number of points. Let $N$ be the smallest integer such that $f^N(y) = f^{N-i}(y)$ for some $i \leq N$, i.e., $N$ is the first time the orbit iterates out a duplicate value. Now, $f$ has only one critical point, $c$. Choose $x_1 \in O(x)$ such that $\forall n \leq N$, $c \in f^N((x_1, y))$. In other words pick a member of $O(x)$ close enough to $y$ so that in one period the interval $(x_1, y)$ does not intersect $c$. Now, $f$ is monotonic under each of the $N$ iterations so all points remain in the same positions relative to each other, and since $O(x)$ converges to $y$, the interval’s length shrinks, giving us $f^N((x_1, y)) \subset (x_1, y)$. So the iteration process maps every $N^{th}$ image as one of a series of nested subintervals of our original interval, and thus never intersect $c$.

Let $\delta_1 = |x_1 - y|$. Since we have assumed the $O(y)$ is periodic or eventually periodic, it is a finite set, so among its members there is a closest pair, call them $a$ and $b$. Let $\delta_2 = |a - b|$. Let $\delta = \min\{\delta_1, \delta_2\}/4$. Next choose $x_2 \in O(x)$ such that $|x_2 - y| < \delta$. Recall that we can get a member of $O(x)$ as close to $y$ as we need because of the convergence. Finally, consider $V = B_{\delta}((a + b)/2)$ and notice that for all $n$, $f^n((x_2, y)) \cap V = \emptyset$, which contradicts the fact that $f$ is topologically transitive.

Thus $O(y)$ must be a dense orbit, and our lemma is proved. \qed
Also by Lemma 5.2.1a, we notice that since \( O(y) \) is dense in some interval, it must be dense everywhere in \( I \).

So, returning to the proof of the theorem, assume that \( I \) has no dense orbit. Then by Lemma 5.2.1a it cannot have an orbit dense on any subinterval and by Lemma 5.2.1b it cannot have any asymptotic orbit. That leaves us with the situation that \( I \) is composed completely of periodic and eventually periodic orbits.

Previously we have shown that there is only a finite number of periodic orbits for a given period for a special class of unimodal functions. By relaxing the requirement that we placed on these functions to include all unimodal functions, we introduce, at most, one more periodic point per period, and we still have a finite number of points per period. In any case, there is a countable number of period lengths, hence a countable number of periodic orbits. We still need to count how many eventually periodic orbits there are.

We have also previously shown that since \( f \) has only one critical point, each iteration fuses at most two points (thus two orbits). So, for any orbit, each iteration can cause only one other orbit to join it. All eventually periodic orbits must join a periodic orbit. We will classify these eventually periodic orbits in the following manner: those that join a periodic orbit and those that join another eventually periodic orbit before joining a periodic orbit.

Those eventually periodic orbits that join periodic orbits are countable. Since the number of iterations is countable, the number of eventually periodic orbits joining a given periodic orbit must be countable.

The rest of the eventually periodic orbits will be counted in the following way. We will call the leading, non-periodic members of an orbit its ‘tail’. This tail has finite length, and all of the
yet-uncounted orbits must join an eventually periodic orbit in its tail. We will fix one of the eventually periodic orbits of the first class and show that the number of eventually periodic orbits that fuse to this fixed orbit is countable. Then we will have a countable collection of countable objects.

So, fix one of the countable eventually periodic orbits of the first class and count the length of its tail; this number corresponds to the maximum number of orbits joining it there. Take the finite number of orbits just counted and count the lengths of all their tails, adding a second tier of orbits. Now count the tail length of all these second-tier orbits getting the finite number of third tier orbits and so on. Thus, for any of the countable eventually periodic orbits that directly join a periodic orbit, there is a countable number of eventually periodic orbits joining it. A countable collection of countable sets, is countable, hence all periodic and eventually periodic orbits are countable in number.

Since all orbits are countable, and they each have a finite number of elements (points), \( I \) is made up of a countable number of points, which is a contradiction to the fact that any subinterval of the real numbers is uncountable. Thus there must be an orbit of \( f \) that is dense in \( I \), and our theorem is proved. \( \square \)

5.3 The Dense Orbit and Periodic Orbits

Lastly, we will show that a dense orbit implies that the periodic points are dense. This theorem shows that Devaney's definition actually can be reduced to only one part, that of topological transitivity, or as the previous theorem showed, the existence of a dense orbit. Note that the following theorem only applies to continuous functions and the version presented only proves the
case concerning unimodal functions, so in the larger setting of some function mapping a metric space into another metric space, the proof of this theorem fails.

The proofs of this theorem and its accompanying lemma are due to Barge and Martin [BM2]. We begin by asserting a preliminary lemma that is, in itself, an interesting fact about the composition of the dense orbit. First we introduce a notation (also due to Barge and Martin [BM2]) used in the proofs.

**Definition 5.3.1:** Let $A_{s,k}(x) = A_{s,k} = \{f^{sn+k}(x) \mid n \geq 0\}$, for $s \geq 1$ and $k \geq 0$.

**Lemma 5.3.1a:** If $f$ is unimodal with a dense orbit, $O(x)$, (in our new notation $A_{1,0}$ is dense) then for all $s \geq 1$ and $k \geq 0$, $A_{s,k}(x)$ is dense in $I$.

In other words, if you look only at every $n^{th}$ element of $O(x)$, you still get a set that is dense in $I$.

**Proof:** Let $O(x)$ be a dense orbit of the unimodal function $f$. Let $s \geq 1$ and $0 \leq r \leq s - 1$. Then define $B_r = \overline{A_{s,r}}$ (noting that $\overline{A}$ means the closure of $A$). Notice that $\bigcup_{r=0}^{s-1} A_{s,r} = A_{1,0}$ as well as $B_0 = A_{1,0} = I$. These facts give us that $\bigcup_{r=0}^{s-1} B_r = I$. Now if the union of a finite number of closed sets is an interval, at least one of those sets must contain an interval. So, there exists an $r$, $0 \leq r \leq s - 1$, such that $B_r$ has a nonempty interior.

We will digress for a moment to mention that if $J$ is a closed interval (not degenerate) then since $f$ is continuous, it must map $J$ to an interval and not a point. Suppose $f$ does map $J$ to a point, say $f(J) = y$. Now, $J$ contains infinitely many members of $O(x)$, which are mapped to $y$, thus $y$ is a member of $O(x)$. Additionally, since there is more than one member of the orbit mapping to the same point, our orbit must be periodic or eventually periodic hence of finite length, contradicting the fact that it is dense.
Returning from our digression, we notice that \( f(A_{s,r}) = A_{s,r-1} \), and \( f(A_{s,s-1}) = A_{s,s} \subset A_{s,0} \).

So, recalling the fact that for \( f \) continuous, \( f(\bar{A}) \subset \bar{f(A)} \), we get that \( f(B_r) \subset B_{r-1} \) and \( f(B_{s-1}) \subset B_0 \). Thus under \( f \), each of the \( B_i \)'s gets mapped into the next, wrapping around at \( r = s-1 \) to \( r = 0 \). Adding this fact to our digression, we find that since one of the \( B_i \)'s has a nonempty interior, all the \( B_i \)'s must have nonempty interiors.

Define \( G \) to be the set of all components of all \( B_i \)'s interiors. Notice that each such component must have a member of \( O(x) \) in it and so can be indexed by the first member of \( O(x) \) to appear in that component. Since there are a countable number of orbit members and a finite number of \( B_i \)'s, \( G \) must be countable, say \( G = \{ g_1, g_2, \ldots \} \). The \( g_i \)'s are a disjoint collection of open intervals. Further note that since the union of all \( \bar{g}_i \)'s is the union of all \( B_i \)'s, which is \( I \), the union of all \( g_i \)'s must be dense in \( I \). Also, since each of the \( B_i \)'s maps into another, \( f(g_i) \subset B_j \).

More importantly, \( f \) must map each \( g_i \) into some component of \( B_j \), because \( f \) maps intervals to intervals. So, there exists an integer, \( k \), such that \( f(g_i) \subset \bar{g}_k \).

Let \( C_i = \bar{g}_i \). The union of all \( C_i \)'s must be \( I \), and any two \( C_i \)'s can intersect only at a common endpoint, if at all. Using the integer, \( k \), found above, we have \( f(C_i) \subset C_k \). This fact tells us that \( f \) maps every \( C_i \) into another.

Appealing to the fact that \( O(x) \) is dense, we can get a point in the interior of each of the \( C_i \)'s to map, within a finite number of iterations, to a point in the interior of any other \( C_i \), including \( C_k \) a second time. Since \( f \) maps \( C_i \)'s into other \( C_i \)'s, we get the existence of an integer, \( j \), such that \( f^j(C_i) \subset C_k \). So, in \( j \) iterations, \( f \) must have mapped \( C_i \) into every other \( C \). Thus, there must be a finite number of \( C_i \)'s, say \( C_1, C_2, \ldots, C_n \).
We claim that for unimodal functions $n = 1$ (noting that in general, $n$ could be 1 or 2). Recall that unimodal functions all have a fixed point at 0, an endpoint of $I$. Since the union of all $C_i$'s is $I$, one of these must contain 0, say $C_1 = [0, a]$. But since 0 is fixed by $f$, we have for all $j > 0$, $f^j(C_1) \cap C_1 \neq \emptyset$ which implies that $f^j(C_1) \subset C_1$, implying that $C_1$ is the only $C$. Thus $n = 1$.

This fact gives us that all the $B_j$'s have the same interior, hence must all be $I$. Thus, $A_{s,r} = I$ for all $0 \leq r \leq s - 1$, meaning that $A_{s,r}$ is dense for all $0 \leq r \leq s - 1$. Since $s$ was arbitrary, we get that $A_{s,k}$ is dense in $I$ for all $k \geq 0$ and $s \geq 1$, and our lemma is proved.

The next theorem follows readily from this lemma.

**Theorem 5.3.1:** Let $f$ be a continuous function. If $f$ has a dense orbit, then the set of periodic points of $f$ is dense in $I$.

**Proof:** Let $f$ be a continuous function and $O(x)$ be a dense orbit of $f$. Let $V$ be an open subinterval of $I$. Since $O(x)$ is dense in $I$, there exists a point, $y \in O(x)$, in $V$. Also, since the orbit is dense, there are points of the orbit greater than $y$, still within $V$. Let $j$, an integer, be such that $f^j(y) \in V, y < f^j(y)$. By Lemma 5.3.1a, $f^j$ is dense in $I$. Let $g = f^j$. Let $l$ be the smallest positive integer such that $g^l(g(y)) < g(y)$. Noting that $g^l(y) = g^{l-1}(g(y)) \geq g(y) > y$ and $g^l(g(y)) < g(y)$, we get that $g^l([y, g(y)]) \supset [y, g(y)]$, which implies that $g^l$ has a fixed point in $[y, g(y)]$. Thus, recalling that $g = f^l$, we get that $f^{kl}$ has a fixed point in $[y, g(y)] \subset V$, hence $f$ has a periodic point in $V$ (fixed points being allowed). Therefore, the periodic points are dense in $I$ and our theorem is proved. 

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6 ~ An Example of the Transition to Chaos

6.1 An Example of Chaos on the Interval

Finally, we will look at a sub-class of the set of transitional families that have a particularly interesting approach to chaos. They are actually chaotic on the entire interval, $I$, at $\lambda_1$. To get this property we will add the following conditions to our transitional family definition:

1) The function, $f$, has a first derivative that is monotonic on the intervals $[0,c]$ and $[c,1]$.

2) Label the non-zero fixed point of $f$ at $\lambda_1$ as point $p$, and label $\hat{p}$, to be the point such that $f_{\lambda_1}(\hat{p}) = p$ then,

   a) the slope of $f$ at $p$ and $\hat{p}$ has a magnitude of 1 or more and
   b) $\hat{p} > |p - c|$.

To begin with we will show that this class of functions begins non-chaotic at $\lambda_0$. In the case of $\lambda_0$, $f(x) = 0$ for all $x$ in $I$, and all points iterate to 0 and remain. Clearly the periodic orbits are not dense on the unit interval, but in fact, they are not dense in any subinterval. Thus, the system fails one of Devaney's conditions for chaos, and this enough to show that a transitional family is not chaotic anywhere in the unit interval at $\lambda_0$.

Here, we will introduce a notational system for the following arguments. Since we are always looking at unimodal functions, every element of the range—except for $f(c)$—corresponds to two elements of the range. More precisely, we have for all $x \in [0,1]\{c\}$, there exists an $\hat{x} \in [0,1]\{c,x\}$ such that $f(x) = f(\hat{x})$. Further, if one of these pairs is a fixed point, we will denote the other member of the pair with the hat.
So, the system starts out non-chaotic, but does it actually demonstrate Devaney’s conditions at \( \lambda_1 \)? The following argument is due to Devaney.[RD]

Observe the function at \( \lambda_1 \), Figure 6. We will label the second fixed point \( p \) and it’s pair \( \hat{p} \). Notice that these two points divide the unit interval into three sub-intervals. A quick analysis of how \( f \) maps the endpoints of each of these sub-intervals and appealing to the continuity of the function we can easily see how \( f \) maps these sub-intervals. We see that \( f \) maps \((0, \hat{p})\) by stretching it onto \((0, p)\); it maps \((\hat{p}, p)\) by folding it in half at \( c \) and mapping the folded sub-interval onto \((p, 1)\); and it maps \((p, 1)\) onto \((0, p)\). The essential property of this action is that, given enough iterations, \( f \) will take any sub-interval, \( J \), and cause it to cover the entire unit interval, \( I \). To formalize this argument, we will use what Devaney called a ‘First Return Map’; the following theorems and lemmas and their proofs are due to Devaney.[RD] To begin, we will need to use a property of the Schwarzian derivative, namely that the negativity of the derivative is preserved under function composition.

**Property 6.1.1:** If the functions \( f \) and \( g \) have negative Schwarzian derivatives, so will their function composition.

**Proof:** Assume \( Sf < 0 \) and \( Sg < 0 \). We need to show that \( S(f \circ g) < 0 \). One simply, but tediously takes the first, second and third derivatives of the composition. Then using the chain rule,
one plugs these expressions into the definition for the Schwarzian derivative Definition 2.4.4. After collecting the proper terms and noting that they are in the form of the Schwarzian derivative, we get 
\[ S(f \circ g)(x) = Sf(g(x)) \cdot (g'(x))^2 + Sg(x). \]
Since \( Sf < 0 \) for all values in its domain, and since the square of any value is positive, clearly the first term is negative. Since \( Sg < 0 \), the second term of the sum is negative.

So clearly, the Schwarzian derivative of the composition is negative.

Corollary 6.1.1: If \( Sf < 0 \), then \( S(f^n) < 0 \).

The corollary follows immediately from repeated application of Property 6.1.1.

Now we will make the construction that Devaney called the First Return Map. Let 
\[ J = [\hat{p}, p), \]
and note that all the functions we are referring to are understood to have their parameter set at \( \lambda_1 \). Observe that \( f^2(J) = [0, p] \supset J \). So, there exists some subset of \( J \) that 'returns' to \( J \) in two iterations. The returning subset will include some interval, \( I_2 = [x_2, p) \), where \( f^2(x_2) = \hat{p} \), such that \( f^2(I_2) = J \). Since \( f \), continuous, maps one-to-one on \( I_2 \) and \( f(I_2) \), it maps \( I_2 \) homeomorphically onto \( J \). Similarly, recalling that our function is unimodal, there must be a matching interval \( \hat{I}_2 = [\hat{p}, \hat{x}_2] \), where \( f^{2}(\hat{x}_2) = \hat{p} \) and \( f^{2}(\hat{p}) = p \), such that \( f^{2}(\hat{I}_2) = J \).

Now, \( f^2 \) must map \((\hat{x}_2, x_2)\) onto \([0, \hat{p})\). Since \( f(x) > x \) for all \( x \in (0, \hat{p}) \) and \( f(\hat{p}) = p \), we have that, under a finite number of iterations, \( x \) will return to \( J \). In fact, in one iteration we have that \( J \subset [a, \hat{p}) \), where \( f(a) = \hat{p} \). This subinterval originated in \( J \) as \([x_3, x_2)\) and \([\hat{x}_2, \hat{x}_3] \). If we label \( I_3 = [x_3, x_2) \) and \( \hat{I}_3 = [\hat{x}_2, \hat{x}_3] \), we note that \( f^3(I_3) = f^3(\hat{I}_3) = J \). Since every \( x \in (0, \hat{p}) \) returns to \( J \) under some finite number of iterations, we can say that if \( x \in J \setminus \{c\} \), then there exists a least integer, \( n \geq 2 \), such that \( f^n(x) \in J \). Let \( \phi(x) \) be this least integer, then we can finish defining the
twin series of intervals by the following: \( I_n = \{ x \in (c, p) | \phi(x) = n \} \) and \( \hat{I}_n = \{ x \in [\hat{p}, c) | \phi(x) = n \} \).

Now we define the First Return Map, \( R: \mathbb{J}\setminus\{c\} \to \mathbb{J} \), by \( R(x) = f^{\phi(x)}(x) \).

Before we use this construction, we will note some of its properties. The discussion will focus on the \( I_n \)'s, but by symmetry, similar arguments and observations hold for the \( \hat{I}_n \)'s.

Observe that since \( R|_{I_n} = f^n|_{I_n} \), for any point, \( x \), in some \( I_n \), \( (f^n)(x) > 0 \) because
\[
(f^n)'(x) = f'(x) \cdot f'(f(x)) \cdot f'(f^2(x)) \cdot f'(f^3(x)) \cdot \ldots \cdot f'(f^{n-1}(x)).
\]
The first two factors of the product are derivatives of points to the right of \( c \), hence their slopes must be negative; the rest are all derivatives of points in \((0, \hat{p}]\), which all have positive slopes. Thus the product must be greater than 0. Similarly for all points, \( x \), in some \( \hat{I}_n \), \( (f^n)'(x) < 0 \). Both these facts give us the foundation to prove the following property.

**Property 6.1.2:** If \( \hat{p} \geq |p - c| \), then \(|R'(x)| > 1\) for all \( x \in J \).

**Proof:** Let \( I_k = [l_k, r_k) \) and \( W_k = \bigcup_{n \geq k} ^{\infty} I_n \). (The proof for the \( \hat{I}_k \) side is done similarly with only a sign change.) Thus, \( W_k = (c, l_k) \). Since \( f^{k}|_{I_k} \neq 0 \), any minimum or maximum of \( f^k(I_k) \) must occur at an endpoint. Since we know that the slope is always positive here, if we can show the slope of the endpoints are both greater than 1, we can use the Schwarzian condition to prove our property. First notice that \( f^k(I_k) = [\hat{p}, p) \), but the length of \( I_k \) is less than half the length of \([\hat{p}, p) \), so somewhere the slope of \( f^k|_{I_k} > 1 \), say at the point \( x_k \in I_k \), \( (f^k)'(x_k) > 1 \).

The length of \( W_k \) is less than the length of \([0, \hat{p}] \). So somewhere the slope of \( f^k|_{W_k} > 1 \), say at the point \( \hat{x}_k \in W_k \), \( (f^k)'(\hat{x}_k) > 1 \). Since \( (f^k)' \) has a negative Schwarzian derivative, it cannot have a positive local minimum. Since \( \hat{x}_k < l_k < x_k \), we notice that \( (f^k)'(l_k) > 1 \). Next we look at \( (f^k)'(r_k) \) and see that \( (f^k)'(r_k) = f'(f^{k-1}(r_k)) \cdot (f^{k-1})'(r_k) \). Observe that \( f^{k-1}(r_k) = \hat{p} \), and that \( (f^{k-1})'(r_k) = (f^{k-1})'(l_{k-1}) \), thus \( (f^k)'(r_k) = f'(\hat{p}) \cdot (f^{k-1})'(l_{k-1}) \). Both factors of the preceding
product are greater than 1. So, $(f^k)'(r_k) > 1$. Therefore, $|R'(x)| > 1$ for all $x$ in $J$ and our property is proved.

This property of the First Return Map allows us to prove that our system has the following critical property of expanding all sub-intervals over $I$.

**Theorem 6.1.1:** If $|R'(x)| > 1$ for all $x$ in $I$, then for any subset $V$ of $I$, there exists an $n$ such that $V \subset f^n(V) = I$.

**Proof:** We will look at this proof in several cases depending on where $V$ is.

Case 1: $V$ is a subset of $J$. This case breaks into two sub-cases.

Sub-case 1a: $V \supset I_k$ (or some $\hat{I}_k$) for some value of $k$. Then $f^k(V) \supset J$, and so $f^{k+3}(V) = I$, completing this sub-case.

Sub-case 1b: $V \subset I_k$. In this sub-case, we appeal to the fact that $f^k$ returns our interval back to $J$, but because the slope of $f^k$ is greater than 1 over all of $V$, the image of $V$ returns to $J$ and is longer than $V$. Since the image of $V$ is a subset of $J$, it must fall under Sub-case 1a or Sub-case 1b. If the former is true, we are done. If the latter, we iterate $f$ until its image is back in $J$, noting that because the slope is always greater than one, each succeeding image will grow without bound. At some point one of the images must overlap more than one of the $I_k$'s, say $I_j$ and $I_{j+1}$. Take $V_0 = V \cap I_j$ and note that $V_1 = f^j(V_0) = [a, b] \subset [I_2, p]$, which makes $V_1$ a subset of the closure of $I_2$. Under iteration, $r^2$ maps the image of $V_1$ back into the closure of $I_2$, only longer than $V_1$, with the right endpoint fixed at $p$. Thus there exists an integer, $m$, such that $f^m(V_1) \supset I_2$ and then $f^{m+2}(V_1) \supset J$, implying $f^{m+2+3}(V_1) = I$. Therefore, $f^{j+m+5}(V) \supset I$, and this case is done.
Case 2: $V$ is not a subset of $J$. Since $|f'(x)| > 1$ for all $x$ not in $J$, there exists a subinterval $V_0 \subset V$ and an integer $n$ such that $V_1 = f^n(V_0) \subset J$. Now, we just apply Case 1 and we are done; the theorem proved.

Now proof of the existence of all three of Devaney’s conditions for Chaos is relatively easy.

The second condition in Devaney’s definition, that of topological transitivity, is immediate. Recall that this condition was defined to be the idea that given any two sub-intervals of $I$, under finite iterations one can be made to overlap the other. Clearly, if $U$ can cover the entire interval in $n$ iterations, then it must intersect with any subset of $I$ within no more than $n$ iterations. Thus, topological transitivity follows.

Though we have previously shown that, for unimodal maps, we actually have already satisfied the other two conditions for Devaney’s definition, showing the density of periodic points is a simple matter; so, we will do so.

We have just shown that for any $J$ in the unit interval there exists a positive integer $n$ such that $f^n(J) \supset J$, which implies that there exists a point fixed by $f^n$, i.e., a fixed point of $f$ or periodic point of $f$ of period $n$ or a factor of $n$. So given any neighborhood of the unit interval, there exists within it a periodic point of $f$. Hence the periodic points of $f$ are dense in the unit interval at $\lambda_1$.

For sensitive dependence on initial conditions we will use Theorem 5.1.1 since the result does not flow immediately from the First Return Map. By Theorem 5.1.1 the topological transitivity and density of periodic points implies that we also have sensitivity to initial conditions at $\lambda_1$.

So, by Devaney’s definition, we have a chaotic system.
7 ~ Conclusion

This paper has covered some of the important theorems, tools, and ideas of one-dimensional, discrete dynamics with particular emphasis on unimodal systems. This field is still quite new and lacks the uniformity of approach that more well-trod areas of mathematics have. As one can see by looking at the various theorems presented, they use methods from various fields. Likewise the definitions the tools have not be standardized.

For example, the very useful method of kneading theory has several incarnations. Devaney’s definition is easy to define initially but has drawbacks in defining the ordering of these sequences. Milnor and Thurston have a method that makes the ordering easier while sacrificing the simplicity of the initial definition of the sequences. They essentially incorporate the odd ordering rules of Devaney’s definition into the actual definition of the sequences resulting in the sequences themselves not being a simple and elegant.

As was mentioned earlier, kneading theory can be broadened to a differentiable system of any number of critical points. If one changes the definition from critical points to turning points, which would include ‘sharp’ turns in the curve, we can use this tool in looking at non-differentiable continuous functions.

Some of the additional diversity in approach to this field can be seen in work by Edson Vargas who takes a measure theory approach to look at relative measures of various sets of periodic points.[EV]

As can be seen by the theorems presented, the important factor in determining the dynamics of a system is the shape of the defining function. For example, the quadratic family of parabolas and an appropriately scaled family of sine functions have essentially the same dynamics because their
general shapes are similar. If one were to deviate from the unimodal shape in a significant way, the
dynamics can change greatly. R. S. MacKay and C. Tresser have done work on bimodal functions,
where the value of the function at 1 is 1, not 0.[TM] This type of system introduces all sort of
pathologies not found in the unimodal case.

This paper also chose to leave out an area of dynamical systems that is a great interest to
some researchers, that of attracting and repelling periodic points. Devany covers some interesting
theorems concerning the existence of attracting periodic points in functions with negative
Schwarzian derivatives.[RD]

Also, work by Zbigniew Nitecki extends from some of our discussion of periodic points and
studies implications of systems whose periodic points form a closed set. He finds that if a system
has a closed periodic set, all points are either wandering or periodic.[ZN]

One of the things that makes this field interesting and challenging to study is that it is
relatively new, and research is heading in all sorts of directions with each researcher generally
coming into the field from another area of mathematics and bringing their own methodologies and
ideas. The field has the feel similar, perhaps, to a frontier town: still rough around the edges and
filled with lots of opportunities to discover new things.
Bibliography


