THESIS TITLE: How Might Teachers Make Use of Student Thinking to Design Instructional Practices That Challenge Students' Misconceptions In Ways That Create Cognitive Conflict and Growth in Understanding?

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This study investigated how teachers might make better use of student thinking to design instructional practices that challenge students' misconceptions in ways that create cognitive conflict and growth in understanding. The primary research questions were: How might teachers design instructional practices in order to achieve understanding? How can cognitive conflict/student thinking promote better mathematical understanding? Is it beneficial for teachers to make use of students' misconceptions to help foster understanding of math concepts?

A case study was conducted with 10 fifth grade students who were identified through a curriculum based pre-assessment test on subtracting mixed numbers with renaming. A post-test was given after five consecutive one-on-one study sessions with each student to determine which strategies promoted student thinking, cognitive conflict, and self-discovery using a variety of instructional methods and tools.

To analyze the data, the researchers examined each study session to understand the thinking patterns that lead to a growth in conceptual understanding. The pre-assessment scores were compared to the post-assessment and trends were identified.

The study exposed an average gain of 45 points from the pre-assessment to the post-assessment. The results made it clear that each instructional tool used by the researcher impacted the success rate. The more teachers focus on student thinking, self-discovery, and studying cognitive conflict, the more opportunities for success.

Keywords: Cognitive Conflict, Fractions, Mathematics Education, Self-Discovery, Student Errors and Misconceptions, & Student Thinking
CHAPTER I

INTRODUCTION

The Problem

Over the past decade, the trend in the United States has indicated that mathematics education is trailing behind its counterparts in other countries. In 1989, the National Education Summit, comprised of present and former governors, former secretaries of education, and public educators and administrators, set forth a vision of improving mathematics and science education (NCES, 1998). One of the six educational goals set forth, outlined the desire for the United States to become a world leader in mathematics and science achievement by the year 2000 (NCES, 1998). According to the 2003 Third International Math and Science Study (TIMSS), U.S. students consistently performed lower than their peers in other participating countries (NCES, 2004). In fact, fourth and eighth grade math students were outperformed by an average of 13 other countries (NCES, 2005). Clearly, the United States did not meet its goal, and no significant improvements have been made since (NCES, 2005).

Fractions: An Area of Concern

In response to the need to improve education, the National Assessment of Educational Progress (NAEP), a national organization that surveys and monitors the accomplishments and progress of U.S. students in selected content areas, developed a national report card in order to track student achievement (NCES, 1998; NCES, 2004). In the 1996 report on the nation's progress, the fourth grade assessment
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segment on fractions, ratios, and proportions was among the most difficult and challenging for students (NCES, 1999). For example, with questions involving representation, equivalence, and ordering of common fractions such as a half or a third, students were showing mastery with only fifty percent accuracy (NCES, 1999). Similar findings were found in the eighth and twelfth grade reports (NCES, 1999). Fractions, ratios, and proportions prove to be a continuous stumbling block for our nation’s math students.

Instructional Practices

Given that fractions, ratios, and proportions continue to be an impediment for many students, we must look at a probable cause to this deficiency. Research has indicated that there is a direct correlation between student achievement and teacher pedagogy (NCES, 1999). The 1996 NAEP report examined the emphasis placed by teachers on certain mathematical processes (NCES, 1999). The data illustrated that teachers place a great deal of emphasis, 92% of their instructional time, on facts and concepts, as well as skills and procedures in both the fourth and eighth grades (NCES, 1999). Little importance was placed on reasoning and communication, of which 42.5% of instructional time was dedicated (NCES, 1999). This lack of emphasis outlines a need to focus on students’ mathematical thinking in order to achieve better mathematical reasoning and communication skills.

Importance of Student Thinking

In order to understand the need to examine student thinking, research was scrutinized within this area. Jean Piaget’s theory of the reinvention of mathematics
Improving Student Understanding encompasses the examination of the role of the teacher, a child's ability to invent their own solutions to problems, and the ability of children to re-invent mathematics for themselves, creating deeper understanding (Piaget, 1975; Piaget, 1973). Traditionally, teachers have attempted to teach mathematics with a modern approach, however using methods considered archaic (Piaget, 1975). These outmoded methods are based primarily on verbal instruction from teacher to student using more procedural rather than conceptual approaches (Piaget, 1975). Piaget believes that in order to create understanding, teachers should allow students to discover their own outcomes, facilitated by the teacher who is conscious that there are many possible conclusions (Piaget, 1973). One way to facilitate this notion is to provide "a multitude of materials available that rise questions in a child's mind without suggesting the answers" (Piaget, 1973, p.23). True understanding is visible when children are given opportunities to discover for themselves true reasons involved in the solution process, thus re-inventing mathematics for themselves (Piaget, 1975). Stepping out of the traditional role, teachers should create learning situations where student curiosity and solution seeking are developed (Piaget, 1975).

**Promoting Student Thinking through CGI**

Piaget's theories have evolved as the anchor for mathematical reform and research. Studies related to student thinking in mathematics are really helping teachers to design more effective instructional practices in ways that develop a much stronger understanding of mathematics (Franke, Kazemi, & Elham, 2001; King, 2002; Zohar & Kravetsky, 2003). One of the innovative instructional practices investigated
was Cognitively Guided Instruction (CGI). CGI focuses on the development of students' mathematical thinking and reasoning (Franke, et al., 2001). Through CGI, teachers make use of, understand, and adapt their lessons to build on their students' knowledge and challenge student thinking (Franke, et al., 2001). CGI helps teachers to better understand their students, thus creating a well-balanced environment conducive to student learning.

Promoting Student Thinking through Cognitive Conflict

Cognitively Guided Instruction is one path teachers are taking to study student thinking. Another avenue teachers are taking is to promote cognitive conflict. Cognitive conflict "arises whenever the learner is aware of discrepancy between the cognitive structure he or she possesses and the cognitive structures (i.e. knowledge and understanding) he or she would like, or needs, to have" (Behr & Harel, 1990, p.83). In other words, when students are presented with a problem they are quick to solve and faced with the dilemma of an opposing solution, a battle of right and wrong begins, known as cognitive conflict. Cognitive conflict is an underlying theme within Piaget's research (Piaget, 1975; Piaget, 1973). For this reason, it is a fundamental part of gaining student understanding of mathematics.

The Need to Study Errors

The peak of cognitive conflict comes when students' misconceptions and errors are challenged (Behr & Harel, 1990). Although errors in our society are often viewed negatively, when used correctly, they can be stepping stones toward greater mathematical understanding (Kazemi & Stipek, 2001). In fact, errors are considered
positive and normal occurrences in the learning process (Ashlock, 2002; Fisher & Lipson, 1986). If errors are used and developed properly, they can create deeper understanding of a particular concept (Fisher et al., 1986). When this concept is recognized and evaluated, informative feedback can be used to encourage dialogue between student and teacher, thus creating cognitive conflict (Fisher et al., 1986).

*Intent of the Study*

In order to improve mathematics education in the United States, as educators, we must first address the needs of our students. The primary purpose of this study was to investigate how teachers might make use of student thinking to design instructional practices that challenges students’ misconceptions in ways that create cognitive conflict and growth in understanding. This study addressed the following questions:

1. How might teachers design instructional practices in order to achieve understanding?
2. How can cognitive conflict/student thinking promote better mathematical understanding?
3. Is it beneficial for teachers to make use of students’ misconceptions to help foster understanding of math concepts?

If students conceptually understand the mathematics they are being taught or are given opportunities to discover understanding for themselves, student achievement in the United States should dramatically shift to a globally competitive ranking.
Design of Study

A case study was conducted with 10 fifth grade students who were identified through a curriculum based pre-assessment test on subtracting mixed numbers with renaming during the course of a fifth grade unit on fractions. The questions on the pre-assessment were compiled by the researchers and classroom teachers based on statements found in literature (Ashlock, 2002). Selected participants met with researchers for a series of five 30-minute study sessions, where previously determined misconceptions were challenged and studied to determine which instructional practices led to a growth in understanding.
CHAPTER II

REVIEW OF LITERATURE

Overview

There is an evident need for mathematics reform in the United States (NCES, 1999; NCES, 1998). According to TIMSS and PISA (Program for International Assessment) studies, we continue to remain behind competing nations in the field of mathematics (NCES, 2001; NCES, 1998). One particular area of weakness is fractions (Steencken & Maher, 2002). According to the NAEP summary of 1996, fractions, ratios, and proportions remains a consistent difficulty for mathematics students, as they are performing with fifty percent accuracy (NAEP, 1999).

In order to achieve significant gains in mathematics achievement, mathematics professionals are encouraged to use a variety of pedagogical methods that thoroughly address student thinking. Of the numerous instructional methods available, CGI and cognitive conflict have proven to be effective teaching tools in the assessment of student thinking (Franke, et al., 2001, Behr & Harel, 1990). Jean Piaget, a respected mathematics activist, provided the foundation for much of the research and philosophies behind student’s cognitive abilities and the use of examining children’s thinking patterns. Piaget’s notion of “re-invention” entails students’ examination of their own errors, which can lead to self-correction, thus creating cognitive conflict (Piaget, 1975). By capitalizing on misconceptions and
errors, teachers can design instructional practices that promote cognitive conflict, which can lead to growth in understanding.

Given the breadth of the topic, a review of relevant literature will take a closer look at the trends in mathematics achievement in the United States, as well as where deficiencies lie within the mathematics realm. In addition, we will also examine the importance of studying student thinking and possible pedagogical tools to remedy low achievement.

*International Studies and the Need for Change*

With the emergence and expansion of the global economy, policymakers and educators have started to look toward international comparisons to gauge our national education system’s performance and progress (NCES, 2003). Such comparisons give comprehensive analyses of the many different aspects of international education systems, focusing on student performance and potential strategies to improve student achievement (NCES, 2003). In order to stay competitive in this progressive economy, the United States has participated in a number of international projects designed to shed light on America’s standing in comparison to other leading countries. For over a decade now, organizations such as the Organization for Economic Cooperation and Development (OECD), the International Association for the Evaluation of Educational Achievement (IEA), OECD’s Program for International Student Assessment (PISA), and the Third International Mathematics and Science Study (TIMSS) have conducted extensive research in a multitude of countries to compare U.S. math and science achievement with that of other participating countries.
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A review of these studies will help to expose student achievement and the evident need for mathematics reform in the United States.

The TIMSS Study. The Trends in International Mathematics and Science Study (TIMSS, formerly known as the Third International Mathematics and Science Study) is one of the largest, most comprehensive, and most rigorous international study of schools and student achievement conducted (NCES, 1998). This study encompasses the assessment of a half-million students in mathematics and science in a number of different grade levels in 41 countries (NCES, 1998). Recent reports document results from the 1995, 1999, and 2003 studies.

Although as of 2003, it appears that over the eight-year progressive study, U.S. mathematics students have seen gains in achievement, much of the data is relative and controversial. In 1995 when U.S. fourth graders were outperforming eighth and twelve graders, it was speculated that curriculum modifications had affected this positive movement (Hoff, 2000). However, when the same fourth graders were reassessed in eighth grade in 1999, their achievement had plummeted to the bottom tier, ahead of only five other countries (Hoff, 2000). These eighth graders, above the international mean in mathematics in fourth grade, proved slightly better than C students on a global curve (Hoff, 2000). The trend that achievement was headed in an upward spiral was quickly dissolved with these results. Additionally, about 9 percent of U.S. eighth graders scored in the top 10 percent of the TIMSS-R international benchmarks in mathematics in 1999. Comparatively in Japan, about 33 percent of eighth graders reached this benchmark (NCES, 2003).
Results from the TIMSS 2003 study have recently been released. A comparison was performed to assess performance changes in the achievement of fourth grade students from 1995 to 2003 (NCES, 2004). In 2003, U.S. fourth grade students scored 518 in mathematics, exceeding the international average of 495, outperforming their peers in 13 of the other 24 participating countries, such as Italy, Australia, New Zealand, and Scotland (NCES, 2004). They performed lower than their peers in 11 countries, such as Singapore, Hong Kong SAR, Japan, and Chinese Taipei (NCES, 2004). In 1995, fourth grade students also scored a 518 comparatively, where no significant increase or decrease could be calculated (NCES, 2004). However, the data indicates that the ranking of U.S. fourth graders relative to their peers in 14 other countries was lower in 2003 than in 1995 (NCES, 2004). In 1995, four countries outperformed U.S. fourth grade students in mathematics, and in 2003, seven countries outperformed fourth graders (NCES, 2004).


Although U.S. eighth grade students made significant gains from 1995 to 2003, several countries whose eighth-graders outperformed U.S. eighth-graders in 1995 suffered declines in their average scores (NCES, 2004). The effects of these
decreases resulted in a higher ranking for U.S. eighth-graders in 2003 (NCES, 2004). Looking at the larger scope, U.S. eighth grade students were outperformed by eighth grade students in 12 countries, on average, and outperformed eighth-graders in 4 countries in 1995 (NCES, 2004). In 2003, U.S. eighth grade students were outperformed by eighth-graders in 7 of these countries, on average, and outperformed eighth-graders in 6 countries (NCES, 2004).

**PISA Results.** Results from the TIMSS studies dramatically outline America’s continuous teetering on the international mean with mathematics and science achievement. Another look at international comparisons was conducted through the Program for International Students Assessment (PISA), a new system of international assessment that focuses on the capabilities of 15-year-olds in the areas of reading literacy, mathematics literacy, and science literacy (NCES, 2001). PISA is sponsored by the Organization for Economic Cooperation and Development (OECD), an intergovernmental organization of 30 industrialized nations that serves as a forum for member countries to cooperate in research and policy development on social and economic topics of common interest (NCES, 2001). To align with this study, the following analysis focuses primarily on mathematics literacy results.

In 2000, 32 countries participated in PISA, including 28 OECD countries and 4 non-OECD countries (NCES, 2001). Results from the 2000 PISA study reveal that the United States’ average does not show a significant difference from the OECD mean in mathematics literacy (NCES, 2001). Mathematics literacy is defined as “the capacity to identify, to understand the role that mathematics plays in the world, to
make well-founded mathematical judgments and to engage in mathematics, in ways that meet the need of an individual’s current and future life as a constructive, concerned and reflective citizen” (NCES, 2001, p.24). In comparison with the other participating countries, the United States’ ranking, in relation to its international counterparts, is approximately the same in both mathematics and science (NCES, 2001). However, compared to scores achieved on the reading literacy scale, more countries outperformed the U.S. in mathematics literacy and science literacy (NCES, 2001). In 2000, the U.S. scored 493 in mathematics literacy, just below the OECD average of 500 (NCES, 2001). Standing just under the international mean, 17 countries scored higher in mathematics literacy, including Japan, Republic of Korea, New Zealand, Finland, Australia, Canada, & Switzerland (NCES, 2001). The U.S. outperformed only 9 countries in mathematics in 2000 (NCES, 2001).

Countries who participated in both TIMSS and PISA offer a unique additional comparison. For example, Australia, Canada, Japan, and Korea scored higher, on average, than the United States in PISA 2000, as well as TIMSS 1999 in mathematics and science (NCES, 2001). This comparison is one example of the nation’s continuous international lag behind other leading countries, despite the age of students and framework differences of TIMSS and PISA studies (NCES, 2001).

**NAEP Results.** With the desire for the United States to compete globally and become a leading country in mathematics and science, the National Assessment of Educational Progress (NAEP), a national organization that surveys and monitors the accomplishments and progress of U.S. students in selected content areas, developed a
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system of tracking student achievement in the U.S., also known as The Nation's Report Card (NCES, 1998; NCES, 2004). Over the past 25 years, NAEP has conducted eight long-term trend assessments to examine growth in mathematics performance of 9-, 13-, and 17-year-old students (NCES, 1998). Overall, the period from 1973 to 1996 showed a positive linear trend for the nation's mathematics students (NCES, 1998). While it seems as though the nation's math students are making slight gains in mathematics achievement when viewed through the scope of the United States, there are still continuous areas of weakness in the mathematics realm. One area that repeatedly stands out as a barrier for students is fractions (NCES, 1999).

Fractions: An Area of Concern. In the 1996 Nation's Report Card, the major proportion of the Number Sense, Properties, and Operations questions that assessed student's ability and knowledge were in the areas of fractions, decimal fractions, percentages, ratios, and proportions (NCES, 1999). Of the numerous sub-sections in mathematics that were measured, these proved most challenging for all three grade levels measured, fourth, eighth, and twelfth grades (NCES, 1999). Questions from this section asked students to identify appropriate representations of common and decimal fractions, order or identify equivalent fractions, and to use their computational skills to solve fractions and percentages problems (NCES, 1999). Four example questions and results will be presented to outline deficiencies in these areas.

Initially, fourth grade questions posed included representation, equivalence, and ordering of common fractions such as a half or a third (NCES, 1999). When
asked to calculate how many fourths make a whole, 50 percent of fourth grade
students were able to answer correctly (NCES, 1999). Another example that
demonstrating weakness in this content area came from the eighth grade portion
centering on percent. Eighth graders were asked to calculate a 15 percent tip on a
$24.99 bill. Of the eighth graders assessed, only 38 percent were able to answer
correctly, an astonishing 62 percent answered incorrectly (NCES, 1999). The third
example problem was given to students in grades 8 and 12. The problem described
the population growth of two towns, both textually and graphically, and gave two
opinions (Brian’s and Darlene’s) regarding the relative growth of the two towns.
Students were asked to use mathematics to explain how either opinion might be
justified (NCES, 1999). Eighth grade results came up with a dramatic one percent
correct, and not far ahead, twelfth graders averaged 3 percent correct overall (NCES,
1999). The final example was asked of twelfth graders only. The problem was
multiple-choice and involved rate and time and tested students’ knowledge of
procedures used to solve for rate per unit of time. Forty-nine percent of the twelfth
graders were able to answer this question correctly (NCES, 1999). All three grade
levels demonstrated significant deficiencies in the areas of fractions, decimal
fractions, percentages, ratios, and proportions.

_Teaching Practices are a Prime Concern._

As a nation, our weaknesses in mathematics continue to handicap our
students’ performance internationally (Dobbs, 2004). We must look for a probable
cause to this deficiency. One theory suggests that students abroad have an edge on
mathematics due to the manner of which mathematics is approached and learned (Hoff, 2000). Students in other countries are more frequently engaged in projects giving them opportunities to discover mathematical concepts and apply them to real-life scenarios (Hoff, 2000). In comparison, the 1996 Nation’s Report Card determined that questions requiring students to solve problems and communicate their reasoning proved most difficult (NCES, 1999). Of the two, communicating their mathematical understanding proved most challenging of all (NCES, 1999).

Additionally, the 2000 PISA study recognized that about half of U.S. 15-year-olds admitted trying to memorize as much as possible often or always when studying (NCES, 2001). Furthermore, many claimed to use memorization as a learning strategy (NCES, 2001). Scrutinizing this data is crucial in creating consensus as to why our nation’s math students are so far behind. The focus now must turn away from the student and narrow its lens on the primary influence of learning, the educator.

Although there are numerous factors that take part in the understanding and achievement of the mathematical mind, teacher practices must be addressed, as they are a primary source of concern in our nation (Bracey, 2000). In the 1996 NAEP study, teachers were asked to determine the extent to which they emphasize the following mathematical processes: learning mathematics facts and concepts, learning skills and procedures to solve routine problems, developing reasoning abilities to solve unique problems, and learning how to communicate ideas in mathematics (NCES, 1999). Together, these mathematical practices provide the foundation for
successful mathematics instruction, as reflected in the NAEP mathematics framework (NCES, 1999).

The 1996 data clearly portrayed a great emphasis by teachers placed on facts and concepts, as well as skills and procedures in both fourth and eighth grades (NCES, 1999). In fact, 92 percent of the fourth grade teachers reported placing “a lot” of emphasis on these two areas. Similar findings were reported from eighth grade teachers whose average was 79 percent. Both reasoning and communication came up short with and average of 45 percent of fourth grade teachers and 48 percent of eighth grade teachers placing a major emphasis in these areas (NCES, 1999).

Knowing facts and concepts is crucial to mathematical development (NCES, 1999). Teachers of both fourth and eighth grade students placed comparable emphasis on learning skills and procedures to solve routine problems as on learning mathematics facts and concepts (NCES, 1999). “In 1996, 91 percent of fourth grade students were taught mathematics by teachers who reported placing “a lot” of emphasis on learning these skills and procedures, whereas 79 percent of eighth grade students had such teachers” (NCES, 1999, p.240).

Although knowing facts and concepts helps set the foundation for mathematical progress, “being able to use one’s knowledge and reasoning ability to solve mathematical problems in contexts that have not been encountered previously” is pivotal in performing mathematics successfully (NCES, 1999, p.241). The 1996 NAEP data suggests that the development of reasoning abilities to solve unique problems is challenging and more often than not, given little attention, with 52
percent of teachers at both grade levels reporting placing “a lot” of instructional emphasis (NCES, 1999). When it comes to communicating ideas in mathematics effectively, 38 percent of fourth grade teachers and 43 percent of eighth grade teachers reported placing an emphasis in their instruction (NCES, 1999). It is important not only for students to be able to solve mathematical problems, but they need to be able to offer explanations and communicate their thought processes and procedures (NCES, 1999).

It is evident from the 1996 NAEP results that teachers are using more procedural methods, emphasizing facts and concepts, and less time is devoted to developing mathematical reasoning and communication skills.

Teacher Practices and the Challenges of Teaching Fractions

The key to achievement in mathematics is through cognitive understanding and being able to apply one’s knowledge to the problem at hand (Franke & Kazemi, 2001). Throughout the eighties, nineties, and now into the twenty-first century, children are consistently having a difficult time grasping the concepts of fractions (NCES, 1998). Many statewide and national assessments have proven time and time again that students do very poorly on the simplest of fraction problems (Hiebert, 1985 & NCES, 1999). Educators are puzzled with the ongoing dilemma in this difficult area. Research has suggested possible solutions to the following questions: What is the root of the problem? How can we bridge the gap to help students succeed? (Hiebert, 1985) The following discussion will shed light on this enduring struggle that educators face when teaching fractions to young children.
It is no secret that fractions appear to be a frustrating area to students and their teachers (Hiebert, 1985). The concept of a fractional piece is not easily interpreted. The interpretation can be altered and fabricated to form many misconceptions in children's minds. Is there a developmentally appropriate age to start introducing fractions? Researchers like Smai Yodintra believe that until a child has mastered the skills involved in logical-thinking, reasoning, creative skills, application and conceptual skills, fractions should not be taught (Yodintra, 1980). In contrast, researchers like Steencken and Maher believe that with the appropriate lessons, explorations, challenging students' thinking and valuable conversations, any child can understand the fraction concepts (Steencken & Mahe, 2002). With opposing research, it is up to the classroom teacher on what stand to take. Fractions will remain a wearisome area if action is not taken.

Bridging the gap is exactly what many educational researchers have set forth to do in the area of fractions (Kamii & Warrington, 1995). In many studies, the procedural based teaching approach verses the conceptual-based teaching approaches are challenged (Steffe, 1990, Zohar & Kravetsky, 2003). Promoting a conceptual understanding of the fraction instead of memorizing rules and procedures outnumbered its' counterparts (Kazemi & Stipek, 2001). Researchers have moved towards more of the inquiry based constructivist approach (Steffe, 1990). In a constructivist classroom, the teacher's role becomes the presenter of tasks in which students will think through a problem and then the teacher will question students about their thinking (Wentworth & Monroe, 1995). Much research has been done
that promotes methods like cognitively guided instruction (CGI), using student thinking, and promoting cognitive conflict as successful teaching tools to close the gap between frustration and student achievement in fractions.

Piaget’s Theory and Practices

In the early writings of Jean Piaget, he sets the stage for research in cognitive development and student thinking. He believes that, “although there is considerable progress in the child’s logical thinking, it is nonetheless still fairly limited” (Piaget, 1975 p.7). It is vital for educators to realize that children’s minds and thinking processes are quite different than those of adults (Piaget, 1975). True understanding manifests itself by self-discovery and it is up to teachers to create opportunities for children to explore and challenge their ideas and thought processes (Piaget, 1975). This solution seeking curiosity will lead children to test out methods and then self-correct themselves (Piaget, 1975). This, in turn, will increase understanding, growth, and development of new mathematical concepts and notions.

In order to properly understand Piaget’s constructivist approach, it is important to discuss the distinctions he has made between the three kinds of knowledge (Kamii & Warrington, 1995). First, physical knowledge is the knowledge of objects in external reality. Simply by observation, physical knowledge can be obtained (Kamii & Warrington, 1995). Secondly, the notion of logico-mathematical knowledge, a state that is not observable knowledge, consists of relationships created by the individual (Kamii & Warrington, 1995). The source of logico-mathematical knowledge is thus in the mind of the child. The third category of knowledge
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according to Piaget is social knowledge (Kamii & Warrington, 1995). An example of this would be words such as “one, two, three”, holidays like Christmas, and the rule of saying “excuse me” when you walk in front of others (Kamii & Warrington, 1995). “Ultimately, the source of social knowledge is conventions worked out by the people” (Kamii & Warrington, 1995 p.59). For children to construct logico-mathematical knowledge, two underlying principles of teaching seem especially important (Kamii & Warrington, 1995). First, it is important to give the right problem at the right time (Kamii & Warrington, 1995, Piaget, 1975). For example, teachers would sequence questions carefully so that children could use what they already knew to invent solutions to more difficult problems (Kamii & Warrington, 1995). Secondly, it is imperative for the teacher to abstain from telling children that an answer is correct or incorrect and instead, encourage them to agree or disagree with each other (Kamii & Warrington, 1995, Piaget, 1975). Piaget said, “Debate is absolutely necessary for children’s construction of objective knowledge” (Kamii & Warrington, 1995). Debating answers leads students to a state of disequilibrium, and cognitive conflict has transpired.

The theories of Piaget are the backbone for studies involving student cognitive conflict, disequilibrium, cognitive dissonance, and inner self-regulation. As Piaget said, “the goal of education is not to turn out people who can only repeat what past generations have done but to produce future generations who are creative, inventive, and capable of doing new things” (Kamii & Warrington, 1995 p. 61). We, as
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educators, must keep this goal in mind every minute of the day, not only in mathematics, but also in every other subject.

*In what ways can teachers promote understanding of fractions?*

**Teaching through CGI.** One innovative teaching method is called CGI, cognitively guided instruction, which is a combination of methods that focuses on high-level cognitive processing, making inferences, drawing conclusions, synthesizing ideas, generating hypotheses, comparing and contrasting, finding and articulating problems, analyzing and evaluating alternatives, and monitoring student thinking (King, 2002). This combination strives to promote mastery of skills and content while challenging students to use their knowledge to draw possible solutions for mathematical problems.

This form of teaching is teacher driven and student run. By posing questions, which lead to discussion among students, teachers are creating an environment conducive to cognitively guided problem solving sessions (King, 2002). Question starters are designed to prompt students to engage in several forms of cognitive activity. These techniques include the “review and consolidation of [student] understanding, checking their comprehension, constructing new knowledge, and monitoring how well they are thinking and learning” (King, 2002 p.35). For example, questions like: What does three fourths look like? Describe three fourths in your own words. These examples would be forms of self- and peer-testing, which would allow students the opportunity to check how well they and their peers understand the material and clarify misunderstandings, correct errors, and fill in the
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gaps of knowledge (King, 2002). At a higher level of thinking, the students are forced into asking and answering thought-provoking questions, forcing students to actually think intensely about the material, reflect back on prior knowledge, and thus, construct new knowledge (King, 2002).

The central goal of CGI is not only to achieve better understanding by students but also to promote multiple connections between new ideas and prior knowledge, creating not only multiple connections but also many different kinds of connections (King, 2002). By doing so, an increase in such cognitive networks will provide cues for recall, which will make the connections more secure and resilient over time, making them easier to remember (King, 2002). As a result, the learning that occurs is above and beyond the material, which takes students to a deeper understanding of the content area (Hiebert & Carpenter, 1992). CGI has proven to be an effective teaching tool in many classrooms today (Hiebert & Carpenter, 1992).

**Promoting Cognitive Conflict.** In conjunction with CGI comes creating a sense of cognitive conflict within students’ minds. Cognitive conflict is defined as: “a perceptual state where one notices the discrepancy between one’s cognitive structure and environment (external information), or between the components of one’s cognitive structure (i.e., one’s conceptions, beliefs, sub-structures and so on which are in cognitive structure)” (Lee & Kwon, 2001 p.5). This instructional strategy has proven to be an easy way to implement higher level thinking skills and promote better understanding (Watson, 2002).
How can teachers engage their students in a state of cognitive conflict? Many have tried a variety of ways and methods. One popular way is by observing the student’s actions and then making a classification of the conflict situations (Behr & Harel, 1990). According to Behr and Harel, there are four ways to identify that a student is quickly approaching a state of disequilibrium. The first is called “missing a fragment,” which entails the student’s cognitive conflict arose because of a missing subprocedure of a correct procedure (Behr & Harel, 1990). This situation may have occurred because the student has either not learned the procedure or has forgotten it (Behr & Harel, 1990). The second is called, “violation of a rule”. This occurs when a student applies an assured procedure that violates a subsequent rule (Behr & Harel, 1990). Either the student has made-up a new method for finding a solution or has reinterpreted the known rule to fit the problem (Behr & Harel, 1990). The third classification is known as “unexpected result” (Behr & Harel, 1990). This situation is frequently seen in classrooms when a student makes a known procedure to a problem, which leads to an unforeseen result (Behr & Harel, 1990). This situation violates the student’s theory and conflict has arisen (Behr & Harel, 1990).

The last category to classify cognitive conflict is through “matching problem components (Behr & Harel, 1990).” This categorization is characterized by the fact that students see familiar workings of one problem and presume that procedures from another problem can be modified and used to suitably solve the problem at hand (Behr & Harel, 1990). With these classifications and categories, educators can
become aware of possible situations where cognitive conflict can be promoted (Behr & Harel, 1990).

Inducing cognitive conflict through inferential reasoning is the key to achieving ultimate success (Watson, 2002). In learning situations, it may occur naturally when one’s guess or hypothesis is proved wrong by a teacher or another classmate, however, for learning to transpire, the conflict must generate dissatisfaction of the original belief (Watson, 2002). The appearance of dissatisfaction or an unapproved technique or explanation causes one to defend or challenge an opposing believer (Watson, 2002). In this respect, active learning is in place, which could result in higher level inferential thinking skills (Watson, 2002). This disequilibrium state will motivate the student to attempt to resolve the conflict (Lee et al., 2002, Kamii & Warrington, 1995, Piaget, 1975, Kamii & Warrington, 1995). If students are engaged and motivated, great strides in their inner self-regulation will occur.

Using Student Thinking to Promote Understanding. Focusing on students’ mathematical thinking remains a dominant method for bringing the gap between pedagogy, mathematics, and student understanding together (Franke & Kazemi, 2001). Using CGI and promoting cognitive conflict are instruments in understanding how children think. Ultimately, educators need to take the time to listen and create opportunities to challenge students’ thinking in order to achieve success (Franke & Kazemi, 2001, Piaget, 1975). By teachers continually evaluating their students’ understanding, adapting and building on their knowledge, and then figuring out how
to use that in the context of their ongoing practice is going to be an essential part of their classroom practices (Franke & Kazemi, 2001). “When individuals learn with understanding, they can apply their knowledge to learn new topics and solve new unfamiliar problems” (Franke & Kazemi, 2001 p102). On the contrary, learners who do not learn with understanding have isolated skills and knowledge and will not make connections from one concept to another (Franke & Kazemi, 2001). Teaching for understanding is the key component to successful students and high achieving classrooms (Hiebert & Carpenter, 1992).

In a recent study by Kazemi & Stipek, they examined ways in which classroom practices press students for conceptual mathematical thinking (2001). The goal of this study was “to describe vividly how teachers can promote student participation in a classroom community where conceptual understanding is valued and developed” (Kazemi & Stipek, 2001 p.60). The outcomes of their study proved that a high-press classroom where students were pushed to go beyond what was easy for them was the most successful (Kazemi & Stipek, 2001). The elemental focus of this inquiry-based mathematics is that students clarify their thinking and push themselves to understanding at a deeper level (Kazemi & Stipek, 2001). This extra push and press is derived by the driving force of the classroom, the teacher.

Why is there a need to study errors?

Teachers have an extremely important and challenging job. With the many teaching methods and strategies like cognitive conflict and CGI, comes the implementation of using students’ errors as a springboard for instruction. Although
errors in our society are often viewed negatively, when used correctly, they can be stepping stones toward greater mathematical understanding. In fact, errors are considered positive and normal occurrences in the learning process (Ashlock, 2002; Fisher & Lipson, 1986). If errors are used and developed correctly, they can create deeper understanding of a particular concept. When this concept is recognized and evaluated, informative feedback can be used to encourage dialogue between student and teacher, thus creating cognitive challenges (Fisher et al., 1986).

Mathematical misconceptions and errors are perplexing in origin and often used interchangeably in literature. In their research, Fisher and Lipson (1986, p.787) define errors and misconceptions. “Errors have been defined as observable behaviors in performance; conceptions are mental constructs. A misconception is therefore not an error but can be an underlying source of error.” Incorrect answers are seldom the result of guessing, low intelligence, or low mathematical ability (Perso, 1992). Errors frequently result from systematic approaches that often have sensible foundations (Perso, 1992).

Nevertheless, educators are well aware of the fact that certain types of mathematical errors are continuously repeated by students (Parish & Ludwig, 1994). Literature shows that students are unaware of these errors and surprisingly, many are brought about by the teacher (Parish & Ludwig, 1994). Since errors are a common occurrence in the classroom, they are expected but not often anticipated by teachers (Perso, 1992). Moreover, teachers do not possess the strategies to remedy these errors and their misconceptions (Perso, 1992). If teachers do not contend with
misconceptions, which can be the cause of errors, students throughout their schooling will suffer anxiety and frustration towards mathematics (Perso, 1992).

*How teachers can use errors to create cognitive conflict.* Cognitive conflict presents itself when the struggle between right and wrong is challenged. When faced with the opportunity to find a solution to a preconceived notion of being correct, a unique situation has arisen. This situation occurs when students are given the opportunity to figure out why they are right or wrong. By challenging misconceptions, questioning what went wrong and why, and providing solid proof of understanding, cognitive conflict is reached.

“One goal of teaching is for students to learn to recognize and correct their own errors. That is, they are to acquire the skills of error management and be able to debug their own [work]” (Fisher & Lipson, 1986). Teachers in the United States often view mathematics as a set of procedures, rather than relationships between concepts (Bracey, 2000). However, it is within this set of procedures where most errors occur.

Responsibility now falls into the teacher’s domain to facilitate an environment conducive to cognitive conflict, where materials are readily available, and by nature, should help facilitate students toward questioning their solutions to posed problems (Newstead, 1999; Steencken & Maher, 2002). Turning the responsibility over to the students, the decision-making process becomes clear, and cognitive conflict has arrived.
Improving Student Understanding

Research suggests that by using students’ errors to teach will provide opportunities to reconceptualize a problem, explore contradictions in solutions, and pursue alternative strategies (Kazemi & Stipek, 2001). Stepping out of the traditional role, teachers should create learning situations where student curiosity and solution seeking are developed (Piaget, 1975). Furthermore, “should the child have difficulties in his attempts to grasp a certain idea, the [appropriate] procedure would not be directly to correct him, but to suggest such counterexamples that the child’s new exploration will lead him to correct himself” (Piaget, 1975, p.9).

Summary

This review of literature suggests that by utilizing different strategies, such as CGI, cognitive conflict, student thinking, and misconceptions, teachers can promote better student understanding, leading to higher achievement. The hope for this study is to encourage fellow teachers to effectively utilize these strategies to induce a richer teaching environment, conducive to a growth in understanding. If teachers are willing to reform their own mathematics instruction to a constructivist one, rich in inquiry-based methods, U.S. students may begin to prevail. International studies have demonstrated the need to reconstruct America’s classrooms. In turn, the U.S. may see a significant increase in the Nation’s goal to compete globally in this progressive economy.
CHAPTER III

METHODOLOGY

The results of international mathematics studies have taken many education professionals by surprise. In response, the drive to improve the student success rate has become a focal area for our Nation’s policymakers. Many researchers feel that there is a solution to this lack of student achievement. This solution may be gained through the use of instructional practices designed to promote cognitive dissonance, which may lead to growth in understanding by our Nation’s students. In order to begin to determine if inquiry-based instructional practices are effective ways to create cognitive understanding, a series of study sessions was conducted. This chapter will describe the research design, selection profiles, instrumentation, procedures, and conclude with how the data was analyzed.

Research Design

A case study was conducted with 10 students who were identified through a curriculum based pre-assessment test on subtracting mixed numbers with renaming during the course of a fifth grade unit on fractions. The questions on the pre-assessment were compiled by the researchers and classroom teachers based on statements found in literature (Ashlock, 2002).

This case study consisted of two phases. In the first phase, students partook in a series of five, thirty-minute one-on-one study sessions with the researchers. In this phase the researchers used techniques to challenge the students’ misconceptions. The second phase consisted of one follow up assessment to determine if any of the
Improving Student Understanding

Instructional techniques helped the students come to realistic solutions to the fraction problems. Each of the 10 students met at the researchers’ and student’s school site before or after school for a series of five, 30 minute study sessions. The aim of this case study was to create a safe environment where students could challenge their mistakes and try to determine other meaningful ways of solving problems. By creating cognitive conflict, the researchers hypothesized that students would move away from their misconception toward the use of more meaningful strategies to correctly solve the fraction problems, as well as future mathematical problems.

Selection Profiles

The participants in this study included 10 fifth grade students from two suburban districts in southern California. Both schools are set in coastal, middle to high socio-economic communities. Of the 10 selected participants, six were female and four were male, ages ranging from nine to eleven.

A pretest was administered to four 5th grade classes to identify which students were making errors while subtracting mixed numbers using renaming (Appendix A). The classroom teachers used this pre-assessment in their fractions unit, as a way of determining how best to teach the unit and to evaluate the certain needs of their students. A total of ten students across all four classrooms were selected based on the pre-assessment for follow-up participation in this fractions study.

Instrumentation

Pre-Assessment: Of the twelve assessment questions (Appendix A), eight were computational and four were one-step word problems using subtraction of
mixed numbers with renaming. As a way of determining the need for each child, no assistance or guidance was given on the pre-assessment.

Study Sessions: Researchers anticipated that the error pattern would be found in students who do not properly rename mixed numbers before subtracting. Once this error pattern had been diagnosed, researchers provided a number of instructional tools to promote cognitive conflict and student thinking. The hope was that students would be able to make sense of the problems better if the researchers offered other tools during the study sessions.

Procedures

Once the participants were chosen, a parent consent form was sent home with each student in order to gain parental permission to conduct research using these students (Appendix C). When all ten students had been granted permission to participate, the study sessions began.

Each student met with one researcher for a series of five, thirty-minute sessions. The first four sessions provided a facet of instructional tools to promote cognitive conflict and student thinking. The tools provided for the students in this study included drawing paper, manipulatives such as fraction circles, base ten blocks, circle and square paper cut outs, and fraction strips.

In addition, realistic problem-based scenarios relevant to students' lives were used. Student strategies were questioned and challenged verbally by the researchers. Questions posed to students were meaningful and academically based to focus on the given content area. These included: “Can you show me another way of solving this
problem? If you had to teach this problem to a friend, how would you do it? How can you prove that this answer is the correct one? How many different ways can you solve this problem?” Real life problems such as sharing pizza at a party or sharing cookies evenly were posed to each student. This created an opportunity to solve these fraction problems in meaningful ways and use this information to compare their previous numerical errors.

The last session consisted of a follow up assessment (Appendices D) to determine if cognitive conflict and student thinking helped the students come to realistic solutions to the fraction problems. The post-assessment consisted of problems similar to the initial assessment. This test helped to determine if the misconception was remedied and which strategies prompted by the researchers were most successful.

Each study session was recorded on audiotape and was transcribed for analysis by the researchers. In addition, specific notes were taken regarding the students’ solution strategies.

Data Processing and Analysis

At the completion of all five study sessions, including post-assessments, data collected was analyzed to determine if mathematical improvement was accomplished. Pre- and post-assessment responses were compared to ascertain which strategy(s) was most successful where misconceptions were remedied. Data tables were created to display and evaluate which instructional tools, decisions, and teacher input strategies
were most successful. The data tables were cross-referenced to create categorical comparisons.
CHAPTER IV

RESULTS

This study investigated how teachers might make use of student thinking to design instructional practices that challenge students' misconceptions in ways that create cognitive conflict and growth in understanding. The purpose of this chapter is to present the results of the study based on the outcomes of the study sessions. The study sessions were conducted in order to identify possible instructional practices that teachers can use to promote understanding of mathematics. The researchers associated trends in student thinking and cognitive conflict in order to create understanding. Additionally, the results will be discussed in a broad sense, focusing on the big picture of the study's outcomes. The results will then narrow to scrutinize each student's individual snapshot.

After all of the data was analyzed, by comparing the results of the pre-assessment to the post-assessment and by looking at the trends of successful and unsuccessful student solution strategies, results were determined. Each study session began with the first problem from the pretest that was scored incorrectly. Students were asked to solve the problem using their preferred method. Initially, the researchers stood back, observing the students' thought processes. As a student progressed through his/her work, the researchers considered the procedure used by the student and which alternative strategy might best produce a favorable result, given the initial attempt was solved incorrectly. Detailed discussions will be laid out with specific rationale for the decisions made by the researchers later in this chapter.
The California Standards Test (CST) scores will be used as a benchmark in grouping students by their previous year's ability levels. The CST measures progress towards California's state-adopted academic content standards, which students should know and be able to do in each grade and subject. The CST scores each student; Advanced (above grade level), Proficient (performing at grade level), Basic (below grade level), Below Basic (below grade level), and Far Below Basic (very below grade level). Table 4.1 lists the pre- and post-assessment results.
Table 4.1  
Pre-Assessment and Post-Assessment Results

<table>
<thead>
<tr>
<th>Students</th>
<th>4th Grade California Standards Test (CST) Results</th>
<th>Pre-Assessment Jan./2005</th>
<th>Post-Assessment Feb./2005</th>
<th>Points Gained or Lost</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Proficient</td>
<td>8%</td>
<td>83%</td>
<td>+75</td>
</tr>
<tr>
<td>B</td>
<td>Basic</td>
<td>8%</td>
<td>58%</td>
<td>+50</td>
</tr>
<tr>
<td>C</td>
<td>Advanced</td>
<td>42%</td>
<td>58%</td>
<td>+16</td>
</tr>
<tr>
<td>D</td>
<td>Advanced</td>
<td>33%</td>
<td>75%</td>
<td>+42</td>
</tr>
<tr>
<td>E</td>
<td>Advanced</td>
<td>42%</td>
<td>42%</td>
<td>0</td>
</tr>
<tr>
<td>L</td>
<td>Advanced</td>
<td>33%</td>
<td>83%</td>
<td>+50</td>
</tr>
<tr>
<td>M</td>
<td>Advanced</td>
<td>25%</td>
<td>92%</td>
<td>+67</td>
</tr>
<tr>
<td>N</td>
<td>Advanced</td>
<td>33%</td>
<td>92%</td>
<td>+59</td>
</tr>
<tr>
<td>O</td>
<td>Advanced</td>
<td>25%</td>
<td>92%</td>
<td>+67</td>
</tr>
<tr>
<td>P</td>
<td>Advanced</td>
<td>17%</td>
<td>42%</td>
<td>+25</td>
</tr>
</tbody>
</table>
The results from the pre- and post-assessments expose a study session average gain of 45 points. Other than one student, Student E, who did not show a gain or a loss, all students improved their scores. Sixty percent of all students had a 50 point or greater gain. The other thirty percent of the students gained between 16 and 42 points. Additionally, 6 out of the 10 students received passing scores of 83% or higher on the post-assessment.

According to the fourth grade CST scores, 80% of the students were considered in the Advanced level of mathematics. Of these students, 7 of the 8 showed a gain, however 3 students did not pass the post-assessment with a 70% or better.

After reviewing the data, it is easy to identify some students who showed tremendous growth from the pre to the post assessment. Students A and B both scored the lowest on the pre-assessment with an 8%. Student A, considered Proficient on the CST, had a significant gain of 75 points from the pre-assessment to the post-assessment, passing the post-assessment with a score of 83%. Also, Student B, ranking Basic on the CST, had a 50 point gain, however did not pass the post-assessment with a 70% or better. In contrast, Student E stood out with no significant gains, scoring the same on both the pre- and post-assessments.

During the study sessions, students were given the opportunity to attempt to resolve the first incorrect problem from the pre-assessment. Students were asked to solve the problem using whichever strategy worked best for them. They were given a variety of instructional tools from which to choose. The tools included drawing
pictures, using fraction strips or circles, circular or rectangular paper cutouts, and base ten blocks. However, students were not required to use these tools. Some students chose to use a traditional approach, such as an algorithm, or attempted a different method, for example, converting the fractions to decimals. The following data table shows which strategies were initially used by the students with an unsuccessful outcome. The results from this table were the initial indicators as to which alternative strategies would later be utilized by the researcher.

Table 4.2
Unsuccessful Preferred Student Strategies

<table>
<thead>
<tr>
<th>Preferred Student Strategy: Unsuccessful Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>Student</td>
</tr>
<tr>
<td>---------</td>
</tr>
<tr>
<td>A</td>
</tr>
<tr>
<td>B</td>
</tr>
<tr>
<td>C</td>
</tr>
<tr>
<td>D</td>
</tr>
<tr>
<td>E</td>
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<tr>
<td>L</td>
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<tr>
<td>M</td>
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<tr>
<td>N</td>
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<tr>
<td>O</td>
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<tr>
<td>P</td>
</tr>
</tbody>
</table>
The results from this table illustrate that 70% of the students chose to attempt the problem by using a procedural based algorithmic approach. Researchers questioned these students to identify the motive behind their choice in strategy. Six out of seven students who chose this approach stated that this strategy was either the quickest method or the way they were taught. Twenty percent of the students chose to draw pictures to help them solve the problem, whereas 10% developed their own strategy of converting the fractions to money or decimals. The two students who drew pictures acknowledged that they enjoyed drawing and were previously exposed to this method. The one student who used decimals made it clear that he felt more confident using currency. Overall, this graph shows that the majority of students chose a procedural based approach to initially solve the fraction problems. It became apparent to the researchers that most of the students would need encouragement to step outside their comfort zone to try a different approach to solve the same problem.

After each student unsuccessfully attempted the fraction problem, the researchers challenged students to solve the same problem using a different method. However, students were unaware that their initial answer was incorrect. Once the problem was attempted for the second time with the new strategy, the student compared their new answer to their previous one. The researchers questioned which response was correct. On their own, students were able to recognize that their method produced the incorrect response, because they were able to reason and justify why their former answer did not make sense. The researchers noted that the math problem became more meaningful once this connection between right and wrong was
established. This debate between right and wrong will be referred to as Cognitive Conflict throughout the remainder of this paper. Jean Piaget defines Cognitive Conflict as the student examination of his/her own errors, which can lead to self-correction, thus creating understanding (Piaget, 1975). Table 4.3 shows which strategies were used by the students with a successful outcome.

Table 4.3
Successful Student Preferred Strategy

<table>
<thead>
<tr>
<th>Student</th>
<th>Drawing pictures</th>
<th>Fraction strips</th>
<th>Fraction Circles</th>
<th>Circular paper cutouts</th>
<th>Rectangular paper cutouts</th>
<th>Base Ten Blocks</th>
<th>Algorithm</th>
<th>Convert to Money/Decimals</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td></td>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>E</td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td>X</td>
<td></td>
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<tr>
<td>L</td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>M</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>N</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>O</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>P</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>X</td>
<td></td>
</tr>
</tbody>
</table>

From this table, it is apparent that a majority of the students, 60%, used circular paper cutouts with a successful result. Thirty percent drew pictures, and 10% used fraction strips with a successful outcome. When students were given the choice
of a second strategy to solve the problem, some were quick to pick a different method, whereas others needed to be coaxed into trying a different way. For the students who needed assistance in choosing an alternate method, the researchers chose for them based upon which strategy fit their learning style. These decisions were based on the students' initial attempt to solve the former problems. If one strategy was too difficult for a student to use, the researcher used a different one. Some trial and error was needed in order to find the best learning modality for each student.

The researchers believe that the circular paper cutouts were the favorable strategy because they are user-friendly and provide a meaningful visual aid to solving the problem. Students were able to show the researchers the steps used in solving the algorithm using the paper cutouts. In addition, when looking back at their new solution, students who used this strategy were able to reason why their new solution was the correct one, because they could prove it by reviewing the steps taken with the circular cutouts.

Researchers noted that cognitive conflict was apparent during these study sessions when comparing the two different solutions by each of the students. All students encountered the disequilibrium between which of the two responses was the correct one. Each student felt challenged to choose which method was correct. Students were overjoyed at their new ability to decipher the correct way to solve a new problem. It was apparent that cognitive conflict produced confidence and a higher sense of achievement.
With the intention of achieving cognitive conflict, researchers needed a path to follow that encouraged students to examine their thinking. Since challenging students to try another technique for solving the same problem would produce disequilibrium, researchers implemented this strategy first. However, not all students were able to rise to this challenge. In order to encourage students to attempt a different way, researchers realized that more questioning of student thought processes needed to occur before the students felt confident to take on a new method. Table 4.4 shows which strategies were implemented by the researchers to challenge the students' initial method of solution, which led to the successful student preferred strategies.

Table 4.4
Instructional Decisions Which Led to Successful Student Strategy

<table>
<thead>
<tr>
<th>Student</th>
<th>Questioning Student Thinking</th>
<th>Challenge student to try another technique</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>B</td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>C</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>E</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>L</td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>M</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>N</td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>O</td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>P</td>
<td>X</td>
<td></td>
</tr>
</tbody>
</table>
From this table, the results show that 60% of the students responded successfully to questioning techniques, which promoted student thinking. For these students, who were unable to select another technique on their own, researchers asked questions regarding their solution methods. This enabled the researchers to better gauge which alternative strategies might assist students to reason through the problem. The researchers considered the following when scaffolding was needed:

1) Was the student making an error, or was the mistake perceived to be true, thus indicating that a misconception had occurred?

2) If students were unable to make sense of the problem, researchers encouraged their thinking by restating the problem using a real-life scenario. This often led to a student-chosen strategy.

3) Instead of showing the students the procedure of solving a fraction problem, it was evident to the researchers that one of the tactile methods needed to be attempted first, in order to have better student understanding of the problem at hand. This often led students to recognize a link between their initial attempt and their second attempt. Then, by comparing and debating two different solutions to the same problem, students were able to better make sense of their solution.

4) After a debate occurred between two different student-generated solutions, the researcher had a decision to make regarding which instructional practices to implement if the student is still unclear as to which solution is the correct one. Students were encouraged to attempt a third method of
solution. The researcher understood that by allowing the student to compare three different strategies and by creating conflict, it would then eventually lead to a more meaningful solution and a higher sense of understanding.

By examining student mistakes, the researchers felt empowered to elicit a strategy to solve the misconception, as opposed to showing them, procedurally, how to solve the problem.

The remaining forty percent of students were challenged to try another strategy that was more meaningful. This population of students was secure in their initial solution method, which produced an incorrect response. Since students were confident with their original methods, researchers challenged them to show their solution again by using a different strategy, thus producing a different outcome. This prompted the debate between which solution was the correct one.

*Looking Deeper.* While looking through the results, a few strategies became more apparent as effective teaching tools. The following table shows the progression of student thinking based on each student's shift of approaches, from unsuccessful to successful, and the gains or loses from the pre- to post-assessment associated with each.
### Improving Student Understanding

#### Table 4.5
Shifts Within Student Strategies and Outcomes

<table>
<thead>
<tr>
<th>Students</th>
<th>Unsuccessful Result Strategy</th>
<th>Successful Result Strategy</th>
<th>Successful Teacher Input Strategy</th>
<th>Points Gained or Lost</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Algorithm</td>
<td>Drawing Pictures</td>
<td>Questioning Student Thinking</td>
<td>+75</td>
</tr>
<tr>
<td>B</td>
<td>Drawing Pictures</td>
<td>Fraction Strips</td>
<td>Challenge student to try another technique</td>
<td>+50</td>
</tr>
<tr>
<td>C</td>
<td>Algorithm</td>
<td>Drawing Pictures</td>
<td>Questioning Student Thinking</td>
<td>+16</td>
</tr>
<tr>
<td>D</td>
<td>Algorithm</td>
<td>Circular Paper Cut-outs</td>
<td>Questioning Student Thinking</td>
<td>+42</td>
</tr>
<tr>
<td>E</td>
<td>Algorithm</td>
<td>Drawing Pictures</td>
<td>Questioning Student Thinking</td>
<td>0</td>
</tr>
<tr>
<td>L</td>
<td>Drawing Pictures</td>
<td>Circular Paper Cut-outs</td>
<td>Challenge student to try another technique</td>
<td>+50</td>
</tr>
<tr>
<td>M</td>
<td>Algorithm</td>
<td>Circular Paper Cut-outs</td>
<td>Questioning Student Thinking</td>
<td>+67</td>
</tr>
<tr>
<td>N</td>
<td>Algorithm</td>
<td>Circular Paper Cut-outs</td>
<td>Challenge student to try another technique</td>
<td>+59</td>
</tr>
<tr>
<td>O</td>
<td>Convert to Money / Decimals</td>
<td>Circular Paper Cut-outs</td>
<td>Challenge student to try another technique</td>
<td>+6</td>
</tr>
<tr>
<td>P</td>
<td>Algorithm</td>
<td>Circular Paper Cut-outs</td>
<td>Questioning Student Thinking</td>
<td>+25</td>
</tr>
</tbody>
</table>
Results from this table indicate that the two instructional strategies implemented by the researchers, “questioning student thinking” and “challenging students to try another technique”, proved to be successful for 9 out of the 10 students. “Questioning student thinking” produced an overall average gain of 37.5 points from the pre-assessment to the post-assessment. However, “challenge students to try another technique” surpassed this total increase with an average gain of 41.25 points.

**Student Samples**

Since each individual participant is unique in their learning style and approach, the remainder of this section will focus on each student’s study sessions. In addition, examples of student work and responses will be included to help clarify results.

According to the fourth grade CST analysis, Student A is considered a Proficient student in mathematics. In this study, she showed the greatest improvement with a score of 8 points on the pre-assessment and an increase to 83 points on the post-assessment, a gain of 75 points overall. Initially, Student A chose to attempt 12 and 1/6 minus 9 and 3/4 using an algorithmic approach. She felt confident with this method and was secure in her response. The researcher assumed that procedural based approaches must have been used throughout her primary years as an initial approach to a series of unfamiliar problems. The researcher then thought that it would be critical for this student to be questioned about her thinking by asking her a series of questions. By questioning the student, the researcher felt it would give
her insight into the student’s thought processes. The following is the dialogue that transpired:

*Researcher: Count out 12 and 1/6 pizzas and then eat or take away 9 and 3/4 of them and see what you get.*

*Student A: Should I draw like normal pizza?*

The researcher then thought the student was making a decision in her mind that she could either use circles or squares, and either would work if they are representative of the fractional pieces.

*Researcher: Well, most of the time pizza comes in a circular form, correct?*

*Student: But, then I will have a hard time dividing the circle into equal slices, could I use square pizzas?*

As the student drew out her pizza squares and drew in lines to represent the fractional pieces, she tried to visualize the solution. The student studied her drawings carefully before coming to any conclusions and voicing her solution. The researcher helped frame the problem by not interrupting and allowing the student to do only what was known for her to do.

She first realized that she needed to cut each pizza into the same number of equal slices to be able to solve the problem. She decided by cutting them into twelve’s she would have an easier time solving the problem with a common denominator. So, she then went back to her paper and changed 1/6 to 2/12 and 3/4 to 9/12. With her square pizza slices, she was able to subtract 9 and 9/12 from 12 and 2/12. She was then able to interpret the pictures and explain the differences between the right and wrong answer she originally came up with using the algorithm. Using
the pictures, she was able to prove exactly how she came up with the correct solution. She was, however, unable to explain her thinking regarding the algorithm. In the remaining study sessions, Student A, found herself talking through problems with the researcher and drawing more pictures to solve problems than using the algorithmic approach. This indicated to the researcher that the student was ready for more complex problems and she needed to connect to the algorithm. The researcher helped the student link her pictures to the algorithm by reviewing each step taken to illustrate the problem. Student A developed a workable strategy to solve problems on her own using the algorithmic approach, resulting in an increase in her post-assessment score and an overall gain of 75 points.

Student B is considered to be at the Basic level in mathematics according to the CST scores in her fourth grade year of schooling. However, during the current school year she is considered to be at risk of retention due to her lack of ability to apply mathematical concepts successfully. Consequently, Student B is in after school tutoring three times a week with a credentialed teacher and also attends private tutoring once a week. In this study, she scored 8% on the pre-assessment and 58% on the post-assessment, a 50 point gain. Student B, loved to draw pictures, however, her connections within fractions and linking her drawings to a correct solution was lacking. The conceptual understanding of part to whole relationships within fractions was unfamiliar to the student. After the researcher realized her situation and ability level, the researcher began to encourage the student to use fraction strips instead of her own drawings to solve the fraction problems. After using the fraction strips and
realizing that the answers were completely different, Student B became very frustrated with the conflict arising between the right and wrong answers. The following dialogue exhibits what emerged between the researcher and the student when attempting to solve problem 9 from the pre-assessment, which involves the subtraction of 5 and 3/8 from 8 (See Sample B):

*Student: I need eight fraction strips to start with.*

The student paused as she examined the different fraction strips available to her. The researcher thought that she might possibly be having a difficult time deciphering which fractional parts to use.

*Researcher: Go ahead and count them out then.*

*Student: But, there are not enough whole ones!*

*Researcher: Well, what can you use to make up strips that are the same size to a whole?*

*Student: I can take the two 1/2 pieces to get a whole.*

Researcher understood that the student had some prior knowledge of equal parts. This demonstrated that the fundamentals of solving fractions by drawing pictures might strengthen her possible success rate in the study sessions. The student then realized her initial problem was with a whole number and a mixed number with eighths as the denominator.

*Student: So I have 8 wholes to begin with. I can take away 5 and 3/8 easier if I do something to one of the whole pieces. I have to make sure that one of my wholes is split into eighths. I know that 8/8 is equal to 1 so that should be o.k.*
Student B verbally explained to the researcher that the answer with the fraction strips was correct and proved it by crossing out the 5 and 3/8 amount that was subtracted from the eight wholes.

Student B worked very hard, trying to connect fraction strips and pictures to algorithmic approaches throughout the remaining study sessions. However, the algorithmic approach still lacked accuracy. Though her success rate was not consistent, Student B managed a 50 point gain from the pre- to the post-assessment.
Sample B.

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<tbody>
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<td></td>
<td>B</td>
<td>1/14/05</td>
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</table>

**Problem:** Tracy is having a birthday party. Her mom ordered 8 pizzas. Since only 10 kids showed up to the party, there were \( 5 \frac{3}{8} \) pizzas left over. How much pizza was eaten?

Taking a better look at Student B's picture and algorithm, the researcher understood that visually, the student could see that her picture solution matched her algorithmic approach. There was little debate, because both solution methods produced the accurate and correct response.
Student C is ranked as an Advanced learner in mathematics in accordance with his CST scores from his fourth grade year of schooling. This student is very vocal and opinionated which allowed him to consistently tell the researcher his way of thinking. The researcher began his first study session with problem 3, 6 minus 1 and 1/4, which he scored incorrect on the pre-assessment (See Sample C1). This problem was challenging for Student C due to the fact that he did not connect the need to borrow and subtract from the whole number. The following is an example of the dialogue that took place:

Student C: I know that this is correct because in fifth grade we do a lot of problems like this.

Student C: I like solving them quick.

The researcher understood by a quick conversation with this student that he felt that mathematics was a surplus of procedures and formulas. Student C’s approach was unsuccessful when he tried the algorithm on his own. When challenged to show the researcher a different way of solving the problem, he chose to draw pictures along with telling a story to help the researcher better understand his thinking. The researcher prompted a story from the student to better understand why he was drawing the pictures in this fashion. The researcher decided quickly that story telling would be a great tool for this student in the remaining study sessions. Student C’s story is as follows:

Student C: O.K., say you have 6 cookies that you are going to eat. That is kind of a lot to eat but anyway. I have six cookies and I eat 1 and 1/4 of them. How many do I have left? Well, if I had to draw out this picture and teach a
friend, I would split all the cookies into fourths and then cross out 1 whole one and 1/4 of another and that would give me an answer of 4 and 3/4 left.

This approach was successful for Student C, and he enjoyed being able to teach the researcher his thoughts about solving this fraction problem. The researcher was delighted to see that he made the connection to divide the cookies into fourths on his own to make it easier to subtract in the end.

During the remaining study sessions, Student C continued to create his own problems to explain to the researcher his method of thinking. Additionally, the researcher challenged Student C to connect his story problems to the algorithm. Student C was able to verbalize the renaming and borrowing techniques with ease and record it on paper (See Sample C2). Student C scored a 42% on the pre-assessment and 58% on the post-assessment, showing a 16 point gain during the course of this study.

Taking a closer look at the reasons to why this particular student did not show much growth could be due to the fact that the post-assessment was done in a quiet non-verbal fashion. No talking was allowed and for Student C, making sense of the problem through stories was his motivating factor. The increase of only 16 points could be a result of his imagination and story telling approach not being used on the post-assessment. However, cognitive conflict was very apparent within these study sessions, because student C was constantly debating his solutions and explaining why he thought they were correct or incorrect.
Sample C1.
Sample C2.

Problem: Tracy is having a birthday party. Her mom ordered 8 pizzas. Since only 7 kids showed up to the party, there were \(5 \frac{3}{8}\) pizzas left over. How much pizza was eaten?

\[
\begin{align*}
\frac{5}{8} & \quad \frac{3}{8} \\
\frac{2}{5} & \quad \frac{5}{8} \\
\hline
\frac{7}{8} & \quad \frac{8}{8} \\
\hline
\frac{2}{8} & \quad \frac{5}{8} \\
\hline
\end{align*}
\]

\( \ast \quad \frac{7}{8} = 8 \)

Student D, an Advanced student in mathematics according to her CST scores, scored 33% on her pre-assessment and 75% on the post-assessment, which showed a 42 point gain during the course of this study. Student D first attempted the fraction
problem by performing a common algorithm. The student felt that her solution was correct, so the researcher encouraged her to try another method. After another strategy was attempted, drawing pictures, a different solution arose. The student began to wonder which answer was correct. She quickly turned to the researcher in astonishment, and looked to the researcher for an answer to her dilemma. The researcher stared back with a puzzled look on her face and began to explain to the student that she must be able to rationalize which answer she believes to be correct. Student D was unable to determine which solution was the correct one. Another method needed to be implemented. The next strategy, changing the problem to a real life context, was motivated by the researcher after the student asked to try a third way to see if two answers out of three would help. During this time, the researcher put the problem into a story about subtracting pizza amounts with the circular paper cutouts to emphasize meaning and bring a real life context to the problem. With these cutouts, the student led the researcher through the problem showing how to subtract mixed numbers with the pizza shaped paper cutouts. Student D examined all three answers and realized that her drawings matched her circular pizza cutouts. The student immediately went back to her initial algorithm and used renaming and borrowing to come to the correct solution.

During Study Session 3 out of five, Student D realized that the algorithmic approach was attainable and began attempting the remainder of the problems in this manner. The researcher believed this was due to Student D’s initial battle between her solutions and the self-discovery period she went through in the preliminary study
Improving Student Understanding 58

sessions. Without prompting from the researcher, she was able to rename the mixed numbers and subtract successfully. However, she felt inclined to double check her responses by using fraction strips to verify the solutions. This method was successful and led the student to better understanding, hence the 42 point gain on the post-assessment at the end of the study sessions.

Student E is truly a unique case in this study. She is the only student who did not improve and show a gain. Her pre-assessment score was 42% and she remained constant with a post-assessment score of 42%. Student E is considered an Advanced student according to her CST scores from fourth grade. She was unsuccessful using algorithms to solve the fraction problems (See Sample E2). When presented with questions from the researcher, Student E began drawing out her solutions, which presented a new challenge to the student, because her answers appeared different than the algorithm (See Sample E1), creating cognitive conflict. She determined that the picture she drew produced the correct answer. She felt using pictures, as a representation to try and solve the remaining problems, would be a successful tool for her. She was also successful in linking the algorithmic approach to her pictures when prompted by the researcher. The researcher asked her to retrace her steps from her pictures and apply this language to the algorithm. Student E was able to explain how to rename the mixed numbers and write it on the algorithm (See Sample E1). After the last study session, the researcher felt very confident about Student E and her ability level. Looking at her post-assessment, even though drawing pictures to solve the problems was her strategy of choice during the study sessions, on her post-
assessment, she did not draw any pictures to help her solve the problems. Instead, she chose to use the algorithmic approach, which proved inaccurate on many problems, scoring a 42%, the same score as on the pre-test. This may have been due to her test anxiety and nervousness during the post-assessment, or she possibly did not want to attempt the time-consuming act of drawing out the pictures. She did not ask any questions, and no assistance was given to her. Another reason for the lack of improvement may be that the student felt insecure using pictures to solve the fraction problems, as it might not look the official and acceptable way to solve formal mathematics problems. This meant she was either rushed and comprehension of the concepts was weak, or she possibly had an off day during the testing window. Student E’s scores were very disappointing to the researcher.
For dessert, Tracy’s mom baked extra large chocolate peanut butter cookies. The cookies were so big, they had to be shared. The kids were able to eat $8\frac{3}{4}$ cookies. How many cookies were left over if Tracy’s mom baked 12 cookies?

\[
\begin{align*}
8\frac{3}{4} & \quad + \quad 3\frac{1}{4} \\
& \quad - \quad 8\frac{3}{4} \\
\hline
& \quad \frac{1}{4}
\end{align*}
\]
Sample E2.

<table>
<thead>
<tr>
<th>Study Session</th>
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<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>E</td>
<td>1/20/05</td>
</tr>
</tbody>
</table>

**Problem:** Tracy is having a birthday party. Her mom ordered 8 pizzas. Since only 10 kids showed up to the party, there were $\frac{53}{8}$ pizzas left over. How much pizza was eaten?

First Attempt:

\[
\begin{align*}
8 & - \frac{3}{8} \\
\hline
7 & - \frac{3}{8} \\
\hline
\end{align*}
\]

Second Attempt:

\[
\begin{align*}
7 & - \frac{3}{8} \\
\hline
5 & - \frac{3}{8} \\
\hline
2 & \frac{5}{8}
\end{align*}
\]

Checking her work:

\[
\begin{align*}
\frac{25}{8} & + \frac{3}{8} \\
\hline
\frac{28}{8}
\end{align*}
\]

Student L, considered an Advanced mathematics student by CST standards, scored 33% on the study’s pre-assessment. She scored 83% on the post-assessment, with a 50 point overall gain. When Student L first began to attempt the first incorrect
Improving Student Understanding

problem from the pretest, she chose to draw pictures. Her pictures resulted in an incorrect solution. The researcher, without letting Student L know she was incorrect, then challenged her to try a different approach of using the circular paper cutouts. It seemed that Student L would be more successful with a visual representation of the problem since she initially attempted a drawing.

This process of cutting out circles and fractional circle pieces produced a different answer than drawing the pictures. Student L determined that the solution from the circular cutouts was the correct one. She was able to explain why the other solutions were incorrect based on the visual examples she could produce from the cutouts. Pushing even further, the researcher challenged Student L to try the problem again in the algorithmic format. Using the visual examples and some questioning from the researcher, she was able to rename the whole numbers and fractions in the algorithms. She solved again and came up with the same solutions.

According the fourth grade CST analysis, Student M is considered an Advanced student in mathematics. On the pre-assessment, her score was 25%, whereas on the post-assessment, she had a gain of 67 points with a score of 92%. During the first study session, her initial attempt to use an algorithmic approach to solve problem 3, 6 minus 1 and 1/4, from the pre-assessment, was unsuccessful. The researcher questioned her thinking by asking her to explain how she came to the solution. She attempted to explain and then determined that her solution was probably incorrect. Sensing that Student M was feeling insecure about her abilities,
the researcher invited her to try the same problem using the manipulatives. The researcher was hoping that using the tactile approach would ease her insecurities.

Student M chose to try the circular paper cutouts to prove her solution. She explained that they seemed the easiest to use. With this visual approach, Student M came up with a successful solution. She identified that her new answer was the correct one by retracing her steps with the paper cutouts. Her entire demeanor shifted to that of a confident math student. When she had realized that she could solve successfully and prove it, she exclaimed, “This is fun!”

Student M was then challenged to put her approach back into the standard algorithmic format. Again, by reviewing the steps she took while solving with the manipulatives, she was able to make sense of the algorithm. She was then able to apply this new knowledge to the remaining study session problems, resulting in correct responses.

Student N, Advanced according to the previous year’s CST results, scored a 33% on the pre-assessment and 92% on the post-assessment, a 59 point gain. When initial problems that were scored incorrect on the pre-assessment were re-visited, Student N chose to attempt to solve the fraction problems by using an algorithmic approach. The first one he attempted was number 2, 8 and 1/3 minus 2/3, from the pre-assessment (see Student Sample N). He attempted to convert or rename 8 1/3 in order to subtract 2/3, knowing that 2/3 is larger than 1/3. However, he was unsure of the procedure. He explained that 8 1/3 should be changed but could not explain or reason a probable solution. The following is the conversation that occurred:
Student N: Ok, I know that 1/3 is smaller than 2/3, so I can’t subtract or it will be negative.

Researcher: So how are you going to subtract?

Student N: Well, I need to borrow like in regular subtraction, but I’m not sure how with a fraction and a whole number.

Researcher: Why don’t you try to borrow and see what you come up with.

Student N: Ok, I’ll borrow from the 8, changing it to 7, but then I need to do something with the fraction. I’ll use the one I borrowed in place of the fraction.

Student N attempted the subtraction by subtracting one value from the numerator. After an incorrect solution of 7 1/3 was reached, the researcher posed the possibility of showing the solution again using manipulatives. Student N chose to try circular cutouts on his own. He chose 9 circles and cut one into thirds. He was able to then cut away 2/3 from another whole circle, leaving 7 wholes and 2 third pieces. The researcher pointed out the two different answers and questioned which one was the correct one. Student N was confident that his second attempt with the circular cutouts was the correct one, because he could look back at it and prove it visually. He was also able to then explain why the first one was incorrect and did not make sense.

After Student N was secure with his final answer, the researcher walked him through the algorithmic approach through a series of questions, linking the circular cutouts to the algorithm. He was able to go back to the algorithm and rename the mixed number, connecting it to the manipulatives.
Problem: \[ \frac{7}{8} + \frac{1}{3} - \frac{2}{3} = \frac{4}{3} - \frac{2}{3} \]

\[ = \frac{2}{3} \]

\[ 7 \frac{2}{3} \]

Student O, considered Advanced on the CST, had a 67 point gain from his pre-assessment score of 25% to his post-assessment score of 92%. When problem 2, 8 and 1/3 minus 2/3, from the pre-assessment was attempted again, Student O chose to convert the fractions to decimals. However, the conversion was not clean and he ended up with a repeating decimal. Even though his decimal answer was correct, the researcher challenged Student O to attempt the problem again using fractions with manipulatives. He chose to use circular cutouts and easily laid out 8 wholes. To create one third, he took one whole, and separated it into thirds with a pencil. He then cut out one third. He realized that he needed to take away two thirds, but would need to break up a whole again. Once he had disposed of the extra one third, he counted
up the remaining wholes and thirds for a total of 7 and 2/3. He confirmed that his response matched his decimal answer.

The researcher chose to take Student O to an algorithmic approach. The following dialogue illustrates the outcome:

Researcher: Let's look back at the original problem, 8 and 1/4 minus 2/3. You easily solved this problem using decimals and circular cutouts. Explain to me what you did.

Student O: I took 8 circles. Then I took one more.

Researcher: Why did you have to take one more?

Student O: So I could have 1/3.

Researcher: How did you show 1/3?

Student O: I cut off 2/3, then I was left with 1/3.

Researcher: Then what did you do?

Student O: I had to take away 2/3, but there was only a 1/3 cut piece. So, I cut out 2/3 from one of the other circles.

Researcher: Ok. Let's look back at the algorithm. You started by laying out 8 whole circles and a 1/3 piece. You just said that you had to cut out 2/3 from one of the whole circles. How did the algorithm change when you did this?

Student O: What do you mean?

Researcher: How did 8 and 1/3 change? Look at your circles.

Student O: Oh, I changed one of the wholes into thirds so I could take away the 2/3.

Researcher: So how many wholes did you end up with before you took away the 2/3?

Student O: There are 7 whole circles on the table.
Researcher: Can you show me on paper how the 8 turned into a 7?

Student O: Can I cross off the 8 and turn it into a 7.

Researcher: Sure. Now what happened to the extra whole?

Student O: I cut it up into thirds.

Researcher: Why?

Student O: So I could take away the 2/3.

Researcher: Before you took away the 2/3, how many thirds did you have?

Student O: I had 1/3 from before and 3/3 from the whole, so I had 4/3.

Researcher: What can you tell me about the value of 4/3?

Student O: It is equal to 1 and 1/3.

Researcher: So how many wholes and thirds did you have before taking away the 2/3?

Student O: 7 wholes and 4/3.

Researcher: Let’s look again at the algorithm. You crossed off the 8 and turned it into a 7, but what did you really do with the 1 whole?

Student O: I divided it up into thirds.

Researcher: Can you show me what that would look like on the paper?
[Student O wrote 3/3 near the 1/3 from the problem. (See Sample O)]

Researcher: So, did the value of 8 and 1/3 change?

Student O: No, because the whole is still there, it is just broken up.

Researcher: Can you show me on paper how many thirds you had total then?
[Student O wrote 4/3 next to the 3/3. (See Student Sample O)]

Researcher: Now that you have 4/3, can you subtract the 2/3?

Student O: Yes.
Researcher: Why?

Student O: Because you can subtract 4 minus 2 and not get a negative number.

Researcher: Can you do the subtraction on paper?
[He wrote 2/3 down and then instantly dropped down the 7. (See Student Sample O)]

Student O: Cool! The answer is what I got with the circles. That was so easy.

Researcher: Do you want to try a few more?

Student O: Yes.

Student O was then able to successfully complete the remainder of the problems with ease using the algorithmic approach. The researcher asked him to prove several of the problems. He began to draw out the circles after several problems, rather than using the circular cutouts, and easily talked his way through them.
Sample O.

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<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1/18/05</td>
</tr>
</tbody>
</table>

Problem: \[ \frac{7}{8} - \frac{2}{3} \]

\[ \frac{21}{24} - \frac{16}{24} = \frac{5}{24} \]

Student P, considered an Advanced mathematics student by CST standards, scored 17% on the study’s pre-assessment. He scored 42% on the post-assessment, with a 25 point overall gain. When Student P first began to attempt problem 3, 6 minus 1 1/4, from the pre-assessment, he chose to try the algorithm but was unsure how to start. After several minutes of silence and no action, the researcher decided that more scaffolding was needed. To create more meaning, the researcher chose to first put the problem into a realistic context. A discussion about pizza followed. Student P was still unsure how to take away the 1/4. He verbally explained that he could take away the 1 whole, but that left him with 5 1/4, which he reasoned was...
possibly correct, but he was still unsure. However, at that point, he still had not written anything down on the paper.

Sensing this student needed more guidance, the researcher encouraged him to use the manipulatives to show the problem again. He chose to use the circular paper cutouts. He laid out 6 whole circles. He quickly removed one to represent the 1 whole that needed to be taken away. After about 30 seconds, he looked at the researcher for assistance. The researcher asked him what he thought he needed to do. Student P decided that he needed to take away $1/4$, but was unsure if he could. The researcher told him he could use the scissors to take away the $1/4$ if that might help. He then divided one whole into fourths and cut out one fourth. When asked what he was left with, he quickly responded with, “$4$ and $3/4$.” The researcher asked him to think back to what he originally said was the answer. Since they were different, the researcher asked him if he could prove which one was most correct. He initially thought that his second answer of $4 3/4$ using the paper cutouts was the correct one, however could not reason why.

Feeling a little uneasy, Student P chose to draw out the 6 circles and cross off $1 1/4$ (See Sample P). He then proclaimed that his answer of $4 3/4$ was the correct one. He was able to then validate why it was correct by taking the researcher through the steps he took. This seemed to ease his frustration, so several more problems were attempted, and he used the circular cutouts to solve them all.

Since Student P seemed to grasp the subtraction of fractions with the manipulatives, the researcher decided to take him back to the original algorithm from
the first problem. This way, Student P could make a connection between visual and procedural. Since many of the later problems required finding common denominators, using the fraction circles presented another obstacle. Once the algorithmic approach could be understood, the hope was that he could apply his new knowledge to solve the remaining problems. After some questioning, he was able to explain how to borrow from the whole and rename it with the fraction 4/4 (See Sample P). After that, subtracting the fraction was effortless.

Sample P.

Student P continued to use the circular cutouts to help him solve most of the remaining problems. However, when asked to try the problems using the algorithm, much prompting was needed each time. When a new study session began, he asked
to look back at his previous work to help him remember what he had done. Although each study session left him feeling confident, he still did not seem to grasp the concept. It seemed as though he still viewed the algorithmic approach as a procedure that he had not yet memorized. The connection between tactile and algorithmic was not made, which could be the reason he did not pass the post-assessment.

These student snapshots provided the researchers with valuable data regarding the importance of examining student thinking and promoting cognitive conflict to achieve mathematical understanding. Research suggests that students’ examination of their own errors, which can lead to self-correction, creates cognitive conflict (Piaget, 1975). This study capitalized on this notion of disequilibrium, which provided encouraging results. In Chapter 5, the focus now must turn away from the student and narrow its lens on the primary influence of learning, the educator, and how this study’s outcomes can be implemented in the mathematics classroom.
CHAPTER V

DISCUSSION

This study aimed at finding effective instructional teaching tools that promote cognitive conflict in order to create mathematical understanding. The goal of our study was to create a springboard of strategies for teachers to utilize when student errors and misconceptions arise in their own mathematics classrooms. By fostering an environment that sets the stage for cognitive conflict, student achievement and understanding are promoted. This discussion will focus largely on responding to the following primary research questions: How might teachers design instructional practices in order to achieve understanding? How can cognitive conflict/student thinking promote better mathematical understanding? Is it beneficial for teachers to make use of students' misconceptions to help foster understanding of math concepts?

Unsuccessful Methods. Research suggests that a majority of teachers are using memorization and procedural based approaches when teaching math concepts (NCES, 1999; NCES, 2001). The 1996 NAEP data clearly portrayed a great emphasis by teachers placed on facts and concepts, as well as skills and procedures in both fourth and eighth grades (NCES, 1999). Additionally, another mathematical study portrayed the use of memorization as a learning strategy for many students (NCES, 2001). We found similar results during our study sessions. The participants in our study used several different methods to initially solve the problems, however, one strategy stood out among the rest. Seventy percent of the students chose to attempt the new problem using a traditional algorithmic approach. The results of this
method proved unsuccessful. Additionally, the students who chose this approach articulated that this method of memorization and procedures was the strategy they were taught. In order to impact and improve future mathematical success rates, instructional practices need to shift from the traditional, procedural based approach to a more progressive approach. This will help to foster better student understanding.

Successful Methods. Moving away from the traditional mathematics classroom, our study discovered several strategies that promoted student achievement. Of the eight strategies available for student use, six out of the ten students in our study chose to use circular paper cutouts to help them solve the fraction equations. It seemed evident that this strategy was used more often since the circles are easy to manipulate and can easily be cut to reflect the fraction problems. In addition, they offer an opportunity for communicating student thought processes and mathematical reasoning. The remaining students in our study, with the exception of one, chose to draw their own circles to make sense of the problems. Both of these methods are similar in design and produced a favorable outcome.

However beneficial the use of the circular cutouts and drawing pictures, these strategies alone may not have produced the successful results obtained had there not been instructor interjections through questioning student thinking, capitalizing on misconceptions, and challenging students to try other techniques. This was the foundation for our study. By questioning student thinking regarding misconceptions and challenging them to provide evidence to proposed solutions, students were engaged in a battle of right and wrong, disequilibrium. This state is the cornerstone
of student understanding. On the pre-assessment, all ten students scored below the 50 percentile. Of the ten participants, however, five had an overwhelming gain of 50 points or greater from the pre- to post-assessment. Additionally, nine out of the ten students showed improvement on the post-test, and six out of the ten scored a 70% or greater. The fact that there was not a decline in test scores suggests that the instructional practices used were successful. These positive results from our study parallel with Piaget’s theory that true understanding manifests itself by self-discovery (Piaget, 1975). He believes that it is up to teachers to create opportunities for children to explore and challenge their ideas and thought processes (Piaget, 1975). In this study we attempted to utilize this notion to create cognitive conflict within the study sessions to achieve better understanding.

Recommendations

The recommendations made from this study are geared toward a classroom environment where the instructor is open and ready to try new instructional methods that may challenge their normal routine. These recommendations are not in any particular order of importance.

The first recommendation is to create an environment where students feel safe to make mistakes, as they are a pivotal point in their learning pendulum. This would allow students to communicate their mathematical reasoning and justify their thinking without feeling inadequate if their response(s) is incorrect. Educators need to be aware that there are no quick fixes in an inquiry-based classroom. An inquiry-based classroom promotes student thinking and examination of student thought processes.
We encourage teachers to model errors and misconceptions as part of the natural learning and discovery process.

Although creating a classroom conducive to studying errors and misconceptions is important, promoting cognitive conflict is also crucial in developing conceptual understanding. Another recommendation is to challenge students to prove their knowledge by attempting problems in multiple fashions. To help students make sense of concepts and make connections to real life, classroom teachers need to provide the opportunity for students to utilize as many different manipulatives and hands-on tools as possible. When a conflict between two solutions arises, encourage students to debate and reason which solution is the correct one. When students correct themselves, they take ownership of the newfound knowledge, and it becomes concrete in their mathematical minds. Therefore, student achievement and understanding will increase.

Future Research

After looking at the results of our study, we feel that teachers who effectively use student errors and misconceptions, while encouraging cognitive conflict, will promote better student understanding. However, we believe that further investigation in a more traditional classroom setting, where the teacher to student ratio is one to 20 or more, will provide a more insightful tool for the field of mathematics educational reform.
References


Appendix A

Fractions Pre-Assessment

1)

\[
7 \frac{1}{2} - 3
\]

2)

\[
8 \frac{1}{3} - \frac{2}{3}
\]

3)

\[
6 - 1 \frac{1}{4}
\]
4) \[ \begin{array}{r}
3 \frac{1}{4} \\
- 2 \frac{3}{4} \\
\end{array} \]

5) \[ \begin{array}{r}
8 \frac{3}{4} \\
- 6 \frac{1}{8} \\
\end{array} \]

6) \[ \begin{array}{r}
5 \frac{3}{8} \\
- 2 \frac{2}{3} \\
\end{array} \]
7) \[9 \frac{1}{5} - 1 \frac{3}{8}\]

8) \[7 \frac{2}{5} - 4 \frac{7}{10}\]
9) Tracy is having a birthday party. Her mom ordered 8 pizzas. Since only 10 kids showed up to the party, there were $5 \frac{3}{8}$ pizzas left over. How much pizza was eaten?

10) For dessert, Tracy’s mom baked extra large chocolate peanut butter cookies. The cookies were so big, they had to be shared. The kids were able to eat $8 \frac{3}{4}$ cookies. How many cookies were left over if Tracy’s mom baked 12 cookies?

11) Jan has read $5 \frac{5}{6}$ books this week. Her goal is to read 8 books by the end of the month. How much more does she need to read to meet her goal?

12) Mike must be $4 \frac{1}{2}$ feet tall in order to ride Viper at Magic Mountain. He is $3 \frac{3}{5}$ feet tall. How much taller does he need to be in order to ride Viper?
Appendix B

Dear Parents/Caretakers,

Katie Euckert, a graduate student at California State University, San Marcos and fourth grade teacher at Pacific Rim Elementary, is conducting a study to identify how teachers can make use of student misconceptions in mathematics in order to improve student understanding through cognitive conflict. Your child is being asked to participate in this study because he/she is in one of the fifth grade classrooms at Pacific Rim Elementary. If you and your student agree to participate in this study, your student will be asked to do the following this trimester:

1. Participate in a series of five after or before school, 30 minute one-on-one study session conducted by the teacher who will pose questions relating to subtraction of fractions. These sessions will be audio taped and transcribed for later analysis.
2. Participate in a post-test given at the completion of the study sessions.

There are no risks in this research greater than those involved in everyday classroom practices and assessment. The potential benefits to your child are that he/she gets individual attention and could receive helpful strategies for continued mathematical understanding. Your participation will also help the teacher to better understand individual student needs and how to best teach to those needs.

Participation in this study is voluntary, and your child may withdraw from the study at any time without penalty. If your student does not participate in this study, his/her performance and grades at school will not be affected in any way. All identification from the data gathered will be coded anonymously so that your child will not be identifiable in the written analysis. Any reference to student study sessions will be anonymous. Notes and audiotapes from the study sessions will be stored in a locked cabinet. After all data has been transcribed, the audiotapes will be properly destroyed. All information gathered in this study can be made available to you upon request. The Carlsbad Unified School District, as well as the Cal State San Marcos Institutional Review Board has approved this study. If you have questions about the study, you may direct those to the teacher/researcher, Katie Euckert at (760) 822-5002, or the researcher’s advisor/professor, Dr. Tom Bennett at (760) 750-4307. Questions about your rights as a research participant should be directed to the Chair of the Board at (760) 750-8820. You will be given a copy of this form to keep for your records. Thank you for your time and consideration.
Check one:
____ I agree to participate in this research study.
____ I do not agree to participate in this research study.

Participant’s name (printed)  Participant’s signature

Parent/Legal Guardian’s signature  Date

Researcher’s Signature
Appendix C

Dear Parents/Caretakers,

Carrie Brewer, a graduate student at California State University, San Marcos and sixth grade teacher at Solana Pacific, is conducting a study to identify how teachers can make use of student misconceptions in mathematics in order to improve student understanding through cognitive conflict. Your child is being asked to participate in this study because he/she is in one of the fifth grade classrooms at Solana Pacific School. If you and your student agree to participate in this study, your student will be asked to do the following this trimester:

1. Participate in a series of five after or before school, 30 minute one-on-one study session conducted by the teacher who will pose questions relating to subtraction of fractions. These sessions will be audio taped and transcribed for later analysis.
2. Participate in a post-test given at the completion of the study sessions.

There are no risks in this research greater than those involved in everyday classroom practices and assessment. The potential benefits to your child are that he/she gets individual attention and could receive helpful strategies for continued mathematical understanding. Your participation will also help the teacher to better understand individual student needs and how to best teach to those needs.

Participation in this study is voluntary, and your child may withdraw from the study at any time without penalty. If your student does not participate in this study, his/her performance and grades at school will not be affected in any way. All identification from the data gathered will be coded anonymously so that your child will not be identifiable in the written analysis. Any reference to student study sessions will be anonymous. Notes and audiotapes from the study sessions will be stored in a locked cabinet. After all data has been transcribed, the audiotapes will be properly destroyed. All information gathered in this study can be made available to you upon request. The Solana Beach School District, as well as the Cal State San Marcos Institutional Review Board has approved this study. If you have questions about the study, you may direct those to the teacher/researcher, Carrie Brewer at (858) 794-4516, or the researcher’s advisor/professor, Dr. Tom Bennett at (760) 750-4307. Questions about your rights as a research participant should be directed to the Chair of the Board at (760) 750-8820. You will be given a copy of this form to keep for your records. Thank you for your time and consideration.
Check one:

____ I agree to participate in this research study.

____ I do not agree to participate in this research study.

---

Participant’s name (printed)  Participant’s signature

Parent/Legal Guardian’s signature  Date

Researcher’s Signature
Appendix D

Fractions Post-Assessment

1) \[ 5 \frac{2}{7} - 4 \]

2) \[ 4 \frac{3}{10} - \frac{7}{10} \]

3) \[ 12 - 7 \frac{3}{8} \]
4) 
\[
\begin{align*}
5 \frac{4}{9} \\
 - 4 \frac{8}{9} \\
\hline
\end{align*}
\]

5) 
\[
\begin{align*}
3 \frac{5}{6} \\
 - 1 \frac{3}{12} \\
\hline
\end{align*}
\]

6) 
\[
\begin{align*}
12 \frac{1}{6} \\
 - 9 \frac{3}{4} \\
\hline
\end{align*}
\]
7) \[
\begin{align*}
8 \frac{1}{4} & - 2 \frac{3}{7} \\
\end{align*}
\]

8) \[
\begin{align*}
10 \frac{2}{5} & - 6 \frac{11}{15} \\
\end{align*}
\]
9) Katie usually drives $7 \frac{5}{12}$ miles on the freeway to work. Sometimes traffic is bad due to weather conditions, and she takes another route, which is 12 miles long. How much shorter is her usual route?

10) Tom had 15 gallons of paint. He used $11 \frac{3}{8}$ gallons for a social studies project. How much paint did he have left over?

11) Scott has run $6 \frac{3}{7}$ miles this week. His goal is to run 20 miles by the end of the month. How many more miles does Scott need to run to meet his goal?

12) Matt must be $5 \frac{2}{3}$ feet tall to ride the go carts at Family Fun Center. He is only $4 \frac{3}{4}$ feet tall. How much taller does he need to be in order to ride the go carts?
Appendix E (CST Sample Questions)

GRADE 4  Math

Which of these is the number 6,005,014?
A five million, five hundred, fourteen
B five million, five thousand, fourteen
C five thousand, five hundred, fourteen
D five billion, five million, fourteen

The estimated cost to build a new baseball stadium is ninety-four million dollars. What is this number in standard form?
A $90,400
B $94,000
C $90,400,000
D $94,000,000

Which of the following has the greatest value?
A 121
B 0.97
C 4.23
D 5.08

What is 67,834,519 rounded to the nearest hundred thousand?
A 67,000,000
B 67,800,000
C 67,830,000
D 67,900,000

Which fraction represents the largest part of a whole?
A \( \frac{1}{6} \)
B \( \frac{1}{4} \)
C \( \frac{1}{3} \)
D \( \frac{1}{2} \)

Which fraction means the same as 0.177?
A \( \frac{17}{100} \)
B \( \frac{17}{1000} \)
C \( \frac{17}{100} \)
D \( \frac{17}{1} \)

The numbers in the pattern decrease by the same amount each time. What are the next three numbers in this pattern?
10, 8, 6, 4, 2, ..., ...
A 0, -2, -4
B 0, -1, -2
C 0, 2, 4
D 0, 1, 2

This is a sample of California Standards Test questions. This is NOT an operational test form. Test scores cannot be projected based on performance on released test questions. Copyright © 2004 California Department of Education.
What fraction is best represented by point P on this number line?

\[ \frac{1}{8} \]

B \[ \frac{1}{5} \]

C \[ \frac{3}{4} \]

D \[ \frac{7}{8} \]

On the number line below, what number does point M represent?

\[ \frac{2}{5} \]

\[ \frac{1}{5} \]

C \[ \frac{7}{10} \]

D \[ \frac{3}{10} \]

There are 88 cases of soda in a warehouse. If there are 24 cans of soda in each case, how many cans of soda are in the warehouse?

A 1392

B 1262

C 1292

D 1292

There are 40 teachers at a school. Each teacher is provided with 2500 sheets of paper. How many sheets of paper is this in all?

A 10,000

B 100,000

C 1,000,000

D 10,000,000

This is a sample of California Standards Test questions. That is NOT an operational test form. Test scores cannot be projected based on performance on released test questions. Copyright © 2004 California Department of Education.
There are 9 rows of seats in a theater. Each row has the same number of seats. If there is a total of 162 seats, how many seats are in each row?

A 17  
B 18  
C 19  
D 20

Which of those is another way to write the product 12 × 7?

A 2 × 3 × 7  
B 3 × 4 × 7  
C 3 × 6 × 7  
D 6 × 6 × 7

Which statement is true?

A The only factors of 8 are 1 and 8.  
B The only factors of 9 are 1 and 9.  
C The only factors of 10 are 1 and 10.  
D The only factors of 11 are 1 and 11.

Which number is represented by \( n \)?

\[ 8 \times n = 128 \]

A 13  
B 14  
C 16  
D 19

What is the value of the expression below?

\[ (13 + 4) - (7 \times 2) \]

A 20  
B 12  
C 10  
D 3

What is the value of the expression below if \( a = 3 \)?

\[ 15 - (a + 8) \]

A 4  
B 12  
C 20  
D 26

What is the value of the expression below?

\[ 5 \times (8 - 2) = \]

A 25  
B 30  
C 38  
D 42
21. Which equation below represents the area (A) of the rectangle in square centimeters?

\[ A = (2 \times 45) + (2 \times 9) \]

A. \( 45 \times A \times 9 \)
B. \( 45 \times 9 \)
C. \( A = (2 \times 45) + (2 \times 9) \)
D. \( 45 \times (2 \times A) + (2 \times 9) \)

22. Look at the problem below.

\[ \square = \Delta + 4 \]

If \( \Delta = 7 \), what is \( \square \)?

A. 3
B. 7
C. 11
D. 14

23. The letters S and T stand for numbers. If \( S - 100 = T - 100 \), which statement is true?

A. \( S = T \)
B. \( S > T \)
C. \( S = T + 100 \)
D. \( S > T + 100 \)

What number goes in the box to make this number sentence true?

\[ (7 - 3) \times 5 = 4 \times \square \]

A. 3
B. 4
C. 5
D. 7

Which statement about the figures is true?

Figure 1

15
4

Figure 2

20
3

A. They both have the same area.
B. They both have the same width.
C. They both have the same length.
D. They both have the same perimeter.
Look at the graph. Point $S$ is at (5, 8). Point $F$ is at (5, 1).

How can you find the number of units from point $S$ to point $F$?
A. Add: $5 + 8$
B. Add: $1 + 8$
C. Subtract: $8 - 5$
D. Subtract: $8 - 1$

Which figures below show pairs of lines that appear to be parallel?

A. Figure 1 only
B. Figure 3 only
C. Figure 1 and Figure 2
D. Figure 2 and Figure 3
Look at the circle with center $O$.

The line segment $AB$ appears to be
A. an arc.
B. a perimeter.
C. a radius.
D. a diameter.

Royce has a bag with 8 red marbles, 4 blue marbles, 5 green marbles, and 9 yellow marbles all the same size. If he pulls out 1 marble without looking, which color is he most likely to choose?
A. red
B. blue
C. green
D. yellow

What is the mode of this set of numbers?
(3, 2, 2, 3, 4, 4, 6)
A. 2
B. 3
C. 4
D. 6