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Inhomogeneous Bond Percolation for Regular Infinite Trees and on a Square Lattice with an N-Periodic Inhomogeneity

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Chapter 1

Percolation on a Graph

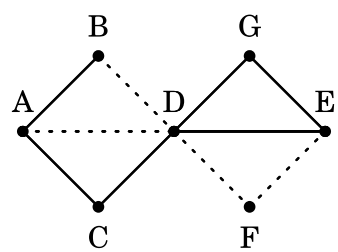

Figure 1.1: Example of a graph with open and closed edges.

A graph consists of a set of vertices and a set of edges; see the example illustrated in Figure 1.1. Vertices are labeled A through G while edges are either drawn as a solid or dotted line segment. We may consider the dotted edges “open” and the solid edges “close”. A path is a sequence of vertices and edges. For example, there is a path connecting A and E in Figure 1.1: beginning at A move to D using the edge that connects them and then move to E using the edge that connects D and E. There is also a path that traverses only open edges, an open path, connecting A and E. Explicitly, beginning at vertex A, move to vertex D using the open edge that connects them, reach vertex F through the open edge connecting D and F, and finally reaching vertex E through the open edge that connects F and E. We call this path an open path. This article concerns graphs with infinitely many distinct vertices, where every pair of vertices is connected by a path. To each edge we associate a probability and determine that that edge to be open randomly with this probability independent of all other edges. Edges not open are claimed to be closed. This produces a random collection of open and closed edges within the graph. We consider the question, what is the probability that a chosen vertex is part of an infinite open path, i.e., a path with infinitely many vertices where all edges are open. This problem falls under the branch of probability called percolation theory.

In Chapter 2, we will start by analyzing tree graphs, to be defined in the next chapter. Each edge will have an associated parameter that determines the probability that the edge is open; for this reason, we sometimes refer to these parameters as probability parameters. We treat two special tree graphs and find closed form expressions for the probability that a chosen vertex is part of an infinite open path. For more general tree graphs, we will derive equations satisfied by this probability, although explicit formulas for the probability that a particular vertex is part of an infinite open path will not be feasible to write or find.

The final chapter will explore N-Periodic inhomogeneous percolation on the square lattice. We consider a multi-parameter model. One parameter is associated with all vertical edges and the other parameters are associated with sets of horizontal edges in a periodic way as described in further detail in Chapter 3. We explore situations in which, by assigning particular values to the probability parameters for the horizontal edges, we are able to determine the so-called critical surfaces (to be defined in Chapter 3) by applying results that are already known for certain special lattices. The cases with just two or three horizontal parameters are explored in some detail. The cases with four or more parameters are more complicated and not treated here.

1.1 Graphs and Associated Definitions

In order to go on further, certain definitions must be formally stated. A *graph* **G** consists of **V(G)**, the set of vertices, and **E(G)**, the set of undirected edges of **G**, and a relation that associates with each edge two vertices which are called *endpoints* of the undirected edge. If e is an edge and v and w are the endpoints of e, then we sometimes use the phrase e is the edge between v and w. Note that there is not necessarily an edge between every pair of vertices in the set **V(G)**. A *subgraph* **G'** of a graph **G** is a graph such that **V(G')** $\subseteq$ **V(G)** and **E(G')** $\subseteq$ **E(G)**. Note that since **G'** is a graph, endpoints of members of **E(G')** are contained in **V(G')**. A *loop* is an edge for which the endpoints are equal. We will only be looking at *simple graphs*, which are graphs in which there are no loops. Given a vertex v in a simple graph **G**, the *degree* of v is the number of distinct edges with v as an endpoint. In addition, the graphs that we consider will have at most one edge associated with any two vertices except for a few exceptions in Chapter 3. This is referred to as the no multiple edges property. Finally, an *infinite graph* is a graph that does not have a finite number of vertices or edges.

When it is clear from context which graph **G** is under consideration, we use the shorthand **V** for **V(G)** and **E** for **E(G)**. A *path* is a sequence $v_0, e_0, v_1, e_1, v_2, e_2, \ldots, v_{k-1}, e_{k-1}, v_k$, where $k \in \mathbb{Z}_+ = \{0,1,2,\ldots\}$, $v_0,\ldots,v_k \in$ **V**,

$e_0,\ldots,e_{k-1} \in \mathbf{E}$, e_i is the edge between v_i and v_{i+1} for $0 \le i \le k-1$ and $e_i \ne e_j$ for $1 \le i \ne j \le k-1$, in which case v_0 and v_k are called endpoints of the path and the path has length k. Given $v, w \in \mathbf{V}$, a path from v to w is a path with endpoints $v_0 = v$ and $v_k = w$. A path may have length zero if the sequence only includes one vertex. Two vertices are *adjacent* if they are endpoints of a common edge; that is, if there exists a path of length one from one vertex to the other. A *circuit* is a path where the endpoints of the path are the same. A graph is called *connected* when there exists a path from any vertex to any other distinct vertex and *disconnected* otherwise. In a connected graph, the *distance* between two vertices is the minimum length over the lengths of all paths connecting the two vertices.

1.2 Percolation on a Graph

Given a graph $\mathbf{G}$, we associate with each edge e a probability p_e, $0 \le p_e \le 1$. We refer to this collection of probabilities as $\mathbf{p(G)}$ or $\mathbf{p}$ when the graph is understood. Edges in a graph $\mathbf{G}$ will have the property of either being 'open' or 'closed'. The determination of whether an edge e is open or closed is made at random and independently of all other edges, but not necessarily in an identically distributed fashion. In particular, an edge e is *open* with probability p_e and *closed* with probability $1 - p_e$. An *open path* in $\mathbf{G}$ is a path where every edge in the path is open. The sample space that we use is $\Omega = \Pi_{e \in \mathbf{E(G)}}\{0,1\}$, which is the set of points $\omega = (\omega(e) : e \in \mathbf{E(G)})$ (called configurations) where $\omega(e) = 1$ when edge e is open and $\omega(e) = 0$ when edge e is closed. *Bond percolation* or just *percolation* for short is the event that there exists a vertex which is connected, via paths of open edges, to infinitely many vertices in the graph. The probability that a particular vertex v is connected to infinitely many other vertices via paths of open edges is called the *percolation probability* of the vertex v and is denoted θ^v. In the following sections, we will be studying *homogeneous* bond percolation of the chosen vertex, meaning all edges in the graph have the same probability of being open, i.e., for some $0 \le p \le 1$, $p_e = p$ for all $e \in \mathbf{E}$. We will also be studying *inhomogeneous* bond percolation, where edges are not necessarily open with the same probability, i.e., p_e may vary with e.

Chapter 2

Percolation on Infinite Regular Trees

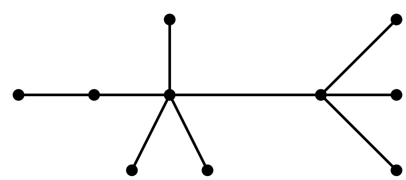

Figure 2.1: Example of a tree graph.

2.1 Tree Graphs

A *tree* is a connected graph with no circuits. In particular, a tree is necessarily a simple graph. A *regular graph* is a graph where every vertex has the same degree. In this thesis, we will be studying infinite $(r+1)-$regular trees, where $r \in \mathbb{N} = \{1,2,3,...\}$. So then, an infinite $(r+1)-$regular tree is a graph that is regular and a tree with an infinite number of vertices. Each vertex of an $(r+1)-$regular tree is an endpoint to $r+1$ distinct edges i.e., each vertex has degree $r+1$. A rooted $(r+1)-$regular tree is an $(r+1)-$regular tree in which one vertex has been distinguished from all others and carries the label *root*. We use the symbol $\emptyset$ to identify it. We will also be studying 'r-ary' trees for $r \in \mathbb{N}$. These are tree graphs with one distinguished vertex called the root (denoted by $\emptyset$) and in which each vertex has degree $r+1$, except for the root, which has degree r. We can obtain an r-ary tree graph as a subgraph of a rooted $(r+1)-$regular tree by deleting one edge that has the root $\emptyset$ as an endpoint, creating a disconnected graph. The connected subgraph of the rooted $(r+1)$-regular tree that contains the root is an r-ary tree. When looking at a pair of vertices in an r-ary tree that are endpoints of an edge, we call the vertex of smaller distance from the root the *parent* and the other the *child*. The distance from the root to the parent is one less than the distance from the root to the child. For $n \in \mathbb{Z}_+$, the *nth generation* is the set of all vertices distance n from the root. A *descendant* v of a vertex u is a vertex such that there is a path from v to u that does not use the edge between u and its parent. Note the root has no parent and all other vertices of the r-ary tree are descendents of the root. We focus on the r-ary rooted tree first and later on place the deleted edge back and consider the full $(r+1)-$regular tree.

2.2 Percolation Models on a r-ary Trees

Due to their translation invariance and lack of circuits, r-ary trees provide a class of examples where computation of percolation probabilities is possible for some $\mathbf{p}$. In fact, for $r = 1$, homogeneous bond percolation at the root only occurs if $p = 1$. Indeed, if $0 \le p < 1$, the probability of percolation at the root is the same event that all the edges of the 1-ary tree are open, which is equal to $\lim_{n\to\infty} p^n = 0$. For $r = 2$ and $r = 3$, the results are less trivial. The 2-ary tree is more commonly referred to as the *binary tree*. Similarly, the 3-ary tree is referred to as the *trinary tree*. Henceforth, we use the notation T_r to denote the r-ary tree for $r \in \mathbb{N}$.

To discuss the result for percolation on T_r, we introduce a vector of probability parameters $\vec{p} = (p_1, p_2, ..., p_r)$ called the *probability vector*. In particular, $0 \le p_i \le 1$ for $i \in \{1,2,...,r\}$. Also we enumerate each edge between a parent and their r children as 1,2,...,r. Then, given a probability vector $\vec{p}$, for each $i \in \{1,2,...,r\}$, p_i is the probability that an edge enumerated by i is open. Let $\hat{\theta}_r(\vec{p})$ be the percolation probability of the root on an r-ary tree with $\vec{p} = (p_1, ..., p_r)$. When $\vec{p}$ is known by the context, we can simplify notation by letting $\hat{\theta}_r = \hat{\theta}_r(\vec{p})$. The *critical surface* is the set of r-tuples $\vec{p}$ that form the boundary of the subset of $[0,1]^r$ such that $\hat{\theta}_r(\vec{p}) = 0$. Equivalently, it is the boundary of the set of $\vec{p}$ such that $\hat{\theta}_r(\vec{p}) > 0$. In the homogeneous case, for some $0 \le p \le 1$ $p_i = p$ for $i \in \{1,2,...,r\}$ (See Figure 2.2).

2.3 Percolation on Binary Trees

Here we consider the simplest nontrivial case of homogeneous bond percolation on the binary tree. So $r = 2$ and $\vec{p} = (p, p)$ for some $0 \le p \le 1$. Then, $\hat{\theta}_2(\vec{p})$ is the percolation probability of the root on the binary tree, where all edges are independently open with probability p. It is easy to see that if $p = 0$ then there is no percolation, and that $p = 1$ will guarantee percolation. We define $p_{c,2} = \sup\{p \ge 0 : \hat{\theta}_2(\vec{p}) = 0 \text{ for } \vec{p} = (p, p)\}$, and call this the *critical probability* for the homogeneous case.

Let Z_2 be the number of vertices in the binary tree connected to the root, and for $n \in \mathbb{Z}_+$, let $Z_{2,n}$ be the number of vertices connected to the root of distance n from the rot. We have $Z_2 = \sum_{n=0}^{\infty} Z_{2,n}$. Hence, by the monotone convergence theorem, $\mathbb{E}[Z_2] = \sum_{n=0}^{\infty} \mathbb{E}[Z_{2,n}]$. For $n \in \mathbb{Z}_+$, there are 2^n vertices of distance n from the root. For $n \in \mathbb{N}$, label those 2^n vertices as 1 through 2^n. Then, for $n \in \mathbb{N}$, $Z_{2,n} = \sum_{i=1}^{2^n} \mathbb{1}_{D_{2,n,i}}$, where for $1 \le i \le 2^n$, $D_{2,n,i} = \{ i^{th}$ vertex at distance n from the root is connected to the root$\}$. For $n \in \mathbb{N}$ and $1 \le i \le 2^n$, $\mathbb{P}(D_{2,n,i}) = p^n$, and so by linearity of expected value $\mathbb{E}[Z_{2,n}] = 2^n p^n$. Since $Z_{2,0} = 1$, we obtain

$$\mathbb{E}[Z_2] = \sum_{n=0}^{\infty} 2^n p^n,$$

which is finite for $p < \frac{1}{2}$. Hence, $\hat{\theta}_2(\vec{p}) = 0$ for $p < \frac{1}{2}$ [2]. So $p_{c,2} \ge \frac{1}{2}$. It is also possible to prove that $\hat{\theta}_2(\vec{p})$ is monotone nondecreasing in p. Therefore, if follows that

$$\begin{cases} \hat{\theta}_2(\vec{p}) = 0, & \text{if } p_1 = p_2 = p < p_{c,2}, \\ \hat{\theta}_2(\vec{p}) > 0, & \text{if } p_1 = p_2 = p > p_{c,2} \left(\ge \frac{1}{2} \right). \end{cases} \tag{2.1}$$

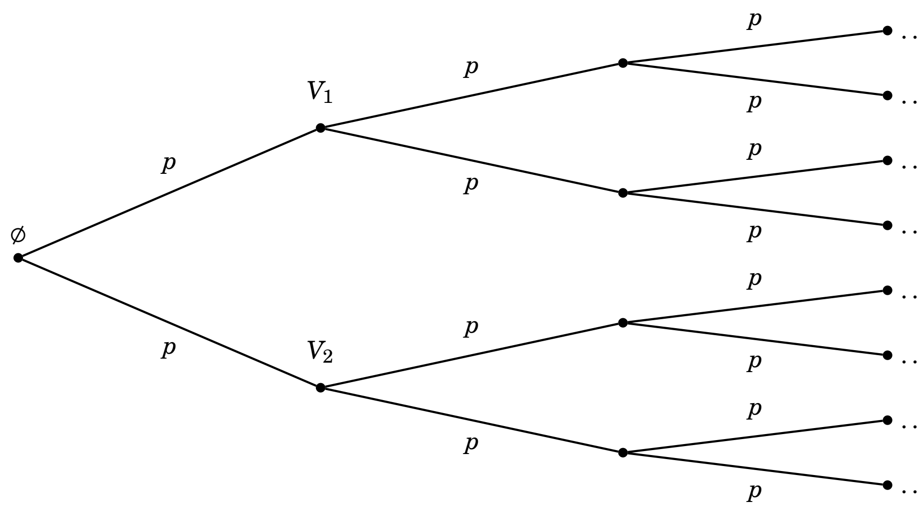

Figure 2.2: Binary Tree with homogeneous edge probabilities.

As we continue to examine the homogeneous case, we ease the notation by writing $\hat{\theta}_2$ for $\hat{\theta}_2(\vec{p})$ where $\vec{p} = (p,p)$ for some $0 \le p \le 1$. We begin by trying to compute $\hat{\theta}_2$. Note that the probability of the root not percolating is $1 - \hat{\theta}_2$. Also, notice from Figure 2.2, if we remove the edge between $\emptyset$ and V_1, a new binary tree is formed that is isomorphic to the original binary tree, where the vertex V_1 is the new root. Thus, the percolation probability of V_1 on this subgraph is equal to $\hat{\theta}_2$, the percolation probability of the root on T_2. The probability that the edge between $\emptyset$ and V_1 is open and V_1 percolating on the subgraph where V_1 is the root is $p\hat{\theta}_2$. Therefore, not percolating on the original graph with root $\emptyset$, on the subgraph with root V_1 is $1-p\hat{\theta}_2$. Similarly, not percolating on the original graph with root $\emptyset$, on the subgraph with root V_2 is $1-p\hat{\theta}_2$. Since these events are independent of each other, this gives the following equalities:

$$1-\hat{\theta}_2 = (1-p\hat{\theta}_2)^2 \tag{2.2}$$

$$= 1-2p\hat{\theta}_2 + p^2\hat{\theta}_2^2. \tag{2.3}$$

Moving the terms to one side, simplifying, and factoring out a $\hat{\theta}_2$ results in

$$\hat{\theta}_2(p^2\hat{\theta}_2 - 2p + 1) = 0.$$

Assume $\hat{\theta}_2 \neq 0$. Then $\hat{\theta}_2 > 0$, $p > 0$ and $p^2\hat{\theta}_2 - 2p + 1 = 0$. We solve for $\hat{\theta}_2$ so that we can write it in terms of p. The result is

$$\hat{\theta}_2 = \frac{2p-1}{p^2}, \text{ if p} \in (p_{c,2}, 1]. \tag{2.4}$$

But $\frac{2p-1}{p^2} \leq 0$ if $0 < p \leq \frac{1}{2}$. So, $\hat{\theta}_2 = 0$ if $0 < p \leq \frac{1}{2}$. Hence, $p_{c,2} \geq \frac{1}{2}$, as was noted in (2.1). If $p_{c,2} = \frac{1}{2}$, it follows from monotonicity of $\hat{\theta}_2$ that

$$\hat{\theta}_2 = \begin{cases} 0, & \text{if } 0 \leq p \leq \frac{1}{2}, \\ \frac{2p-1}{p^2}, & \text{if } \frac{1}{2} < p \leq 1. \end{cases} \tag{2.5}$$

However, one must show that $p_{c,2} \leq \frac{1}{2}$ to substantiate this, which we do in the next section.

2.3.1 Homogeneous Percolation

We use Branching Process Theory [4] to demonstrate that $p_{c,2} \leq \frac{1}{2}$. First, we define some random variables. For $n \in \mathbb{Z}_+$, recall that $Z_{2,n}$ be the total number of all vertices in the binary tree of distance n from the root that are connected to the root by an open path. We call the union of all of the events $\{Z_{2,n} = 0\}$, as n ranges over $\mathbb{Z}_+$, the extinction event A_2 for the binary tree, i.e., $A_2 = \bigcup_{n=0}^{\infty} \{Z_{2,n} = 0\}$ and note that its complementary event $A_2^C = \bigcap_{n=0}^{\infty} \{Z_{2,n} \neq 0\}$ is the event that there is percolation from the root. Also note that, if $Z_{2,n} = 0$ for some $n \in \mathbb{Z}_+$, then $Z_{2,m} = 0$ for all $m \geq n$, meaning that $\{\{Z_{2,n} = 0\} : n \in \mathbb{Z}_+\}$ is an increasing sequence of events. If we write ε_2 for the probability of A_2, then $\varepsilon_2 = 1 - \hat{\theta}_2$. From (2.4), it follows that $\varepsilon_2 = \left(\frac{1-p}{p}\right)^2$ for $p > p_{2,c}$ and $\varepsilon_2 = 1$ for $p < p_{2,c}$. In what follows, branching process theory will be used to determine which value ε_2 takes.

For $j = 0,1,2$, let $\varphi_2(j)$ be the probability that exactly j of the two edges between v and its two children are open. Thus $\varphi_2(0) = (1-p)^2$, $\varphi_2(1) = 2p(1-p)$, and $\varphi_2(2) = p^2$. If $p = 1$, then $\varphi_2(0) = 0$ and $\varepsilon_2 = 0$. Also if $p = 0$, then $\varphi_2(0) = 1$, and $\varepsilon_2 = 1$. Henceforth, we assume $0 < p < 1$. Thus, $0 < \varphi_2(0) < 1$ and $\varphi_2(0) + \varphi_2(1) < 1$, i.e., $\varphi_2(2) > 0$.

Let the probability generating function of $\varphi_2(j)$, $j = 0,1,2$, be given by

$$\Phi(s) = \sum_{k=0}^{2} \varphi_2(k)s^k, \qquad s \in \mathbb{R}.$$

When the probability ε_2 is plugged into the $\Phi(s)$, note

$$\Phi(\varepsilon_2) = \sum_{j=0}^{2} \varphi_2(j)(\varepsilon_2)^j = (1-p)^2 + 2(1-p)p\varepsilon_2 + p^2\varepsilon_2^2. \tag{2.6}$$

The three terms on the right hand side of (2.6) represent the probability of not percolating from the root in three different scenarios and we claim that $\Phi(\varepsilon_2) = \varepsilon_2$. To see this observe that when we condition the occurrence of A_2 on the number of edges between $\emptyset$ and its children, we find that the conditional probability of A_2 occurring is 1 if neither edge is open (which occurs with probability $(1-p)^2$). The second term is the probability that one edge emanating from the root is closed and the other is open, but the root does not percolate. Let's say that the edge between $\emptyset$ and V_1 is closed and $\emptyset$ and V_2 is open (See Figure 2.2). Since the edge between $\emptyset$ and V_1 is closed, there can only be percolation from $\emptyset$ if there is percolation from V_2 on the isomorphic subgraph that has V_2 as its root. And the probability that V_2 doesn't percolate on this subgraph is ε_2. So we obtain $p(1-p)\varepsilon_2$. By reversing the roles of V_1 and V_2, we obtain a second term of this form, hence the factor of 2 in the second term of (2.6). The third term is the probability that both edges are open, but no percolation occurs through either V_1 or V_2. Using a line of reasoning similar to what we argued above for the second term, we have that for this probability is $(p\varepsilon_2)^2$. The sum of these terms gives the probability of $\{Z_{2,n} = 0 \text{ for some } n\}$. Thus,

$$\Phi(\varepsilon_2) = \varepsilon_2. \tag{2.7}$$

This argument should look familiar since (2.6) and (2.7) are effectively a reformation of (2.2) and (2.3). Specifically,

$$\begin{aligned}
1-\hat{\theta}_2 = \varepsilon_2 = \Phi(\varepsilon_2) &= (1-p)^2\varepsilon_2^0 + 2p(1-p)\varepsilon_2^1 + p^2\varepsilon_2^2 \\
&= (1-p)^2(1-\hat{\theta}_2)^0 + 2p(1-p)(1-\hat{\theta}_2)^1 + p^2(1-\hat{\theta}_2)^2 \\
&= 1-2p+p^2+2p-2p^2-2p\hat{\theta}_2+2p^2\hat{\theta}_2+p^2-2p^2\hat{\theta}_2+p^2\hat{\theta}_2^2 \\
&= 1-2p\hat{\theta}_2+p^2\hat{\theta}_2^2 = (1-p\hat{\theta}_2)^2.
\end{aligned}$$

So the probability of extinction or equivalently the probability of not percolating, is a fixed point of $\Phi(s)$. Therefore, we are naturally interested in fixed points of $\Phi(s)$. Furthermore since $0 \le \varepsilon_2 \le 1$, we are interested in fixed points of $\Phi(s)$ in $[0,1]$. For this case, we can can compute the fixed points explicitly by using the quadratic formula. This yields $\varepsilon_2 = \left(\frac{1-p}{p}\right)^2$ or $\varepsilon_2 = 1$, as previously noted. However, if $p > \frac{1}{2}$, both options are fixed points in $[0,1]$, and we need to determine which is the extinction probability.

Observe that the derivative of our generating function is given by

$$\Phi'(s) = \sum_{k=1}^{2} k\varphi_2(k)s^{k-1} = \varphi_2(1) + 2\varphi_2(2)s, \qquad s \in \mathbb{R}.$$

For $s > 0$, $\Phi'(s) > 0$, because $\varphi_1(1) + \varphi_2(2) > 0$. If we evaluate $\Phi'(s)$ at $s = 1$, this gives us the expected number of open edges between a parent and their children. We let $\mu = \Phi'(1)$ so

$$\mu = \sum_{k=1}^{2} k\varphi_2(k).$$

Also note, since $\varphi_2(2) > 0$,

$$\Phi''(s) = 2\varphi_2(2) > 0.$$

Together these facts tell us that $\Phi(s)$ is strictly increasing on $[0,\infty)$ and it is a strictly convex function on $[0,\infty)$, meaning any line segment joining any two points on the curve $y = \Phi(s)$ is strictly above the curve of $\Phi(s)$ except at its endpoints. Mathematically, $\forall x, y \ge 0$ and $0 \le \lambda \le 1$,

$$\Phi(\lambda x + (1-\lambda)y) \le \lambda\Phi(x) + (1-\lambda)\Phi(y).$$

We know that the extinction probability ε_2 is a fixed point of $\Phi(s)$ in $[0,1]$. With this in mind, we now study the fixed points of this strictly increasing, convex function $\Phi(s)$ in $[0,1]$ to determine when ε_2 is not equal to 1.

Notice, $\Phi(0) = \varphi_2(0) > 0$, so that $s = 0$ is not a fixed point of $\Phi(s)$. Also, $\Phi(1) = \sum_{k=0}^{2} \varphi_2(k) = 1$, so that $s = 1$ is a fixed point of $\Phi(s)$. In the next paragraphs, it will be shown that $\Phi(s) = s$ has exactly one solution ($s = 1$) if

$\mu \le 1$ and exactly two solutions in $[0,1]$ if $\mu > 1$. Furthermore, it will be shown that, in case $\mu > 1$, the smallest positive solution is the extinction probability ε_2. In particular, we will be showing that $\varepsilon_2 = 1$ if $\mu \le 1$ and $0 < \varepsilon_2 < 1$ if $\mu > 1$.

Although Φ is a quadratic function in the case of the binary tree, we shall present these arguments in a more general form so that they will also apply to r-ary trees. Specifically, we shall only make use of the following assumptions:

(i) Φ is continuous on [0,1]

(ii) $\Phi(0) > 0$

(iii) Φ is twice-differentiable on (0,1)

(iv) Φ'' is positive on (0,1).

First, let us consider $\mu \le 1$. We know $\Phi(0) = \varphi_2(0) > 0$ and so $s = 0$ is not a fixed point. We also know $\Phi(1) = 1$ and so $s = 1$ is a fixed point. So it suffices to consider $\Phi(s)$ on $(0,1)$. Suppose that $\Phi(s) = s$ for $s = s_0$ for some $s_0 \in (0,1)$. By the Mean Value Theorem, there exists $s_1 \in (s_0, 1)$ such that

$$\Phi'(s_1) = \frac{\Phi(1) - \Phi(s_0)}{1 - s_0} = \frac{1 - s_0}{1 - s_0} = 1.$$

Because $\Phi'(s)$ is strictly increasing, due to convexity on $(0,1)$, we have that $1 = \Phi'(s_1) < \Phi'(1) = \mu \le 1$, or $1 < 1$, which is a contraction. So, when $\mu \le 1$, $\Phi(s) \ne s$ for all $s \in (0,1)$. Thus, $s = 1$ is the unique fixed point on $[0,1]$. Therefore, if $\mu \le 1$, $\varepsilon_2 = 1$.

Next, we assume $\mu > 1$, i.e., $\Phi'(1) > 1$. Since $\Phi''(s) > 0$ on $(0,1)$, $\Phi'(s)$ is strictly increasing on $[0,1]$. Then since $\Phi'(s)$ is continuous, this implies that there must exists a $\delta > 0$ such that $\Phi'(s) > 1$ on $(1-\delta, 1)$. Suppose for some $s_2 \in (1-\delta, 1)$, $\Phi(s_2) \ge s_2$. By the Mean Value Theorem, this means that there exists $s_3 \in (s_2, 1) \subseteq (1-\delta, 1)$ such that

$$\Phi'(s_3) = \frac{\Phi(1) - \Phi(s_2)}{1 - s_2} = \frac{1 - \Phi(s_2)}{1 - s_2} \le 1.$$

But $s_3 \in (s_2, 1) \subseteq (1-\delta, 1)$. So $\Phi'(s_3) > 1$, which contradicts $\Phi'(s_3) \le 1$. Therefore, $\Phi(s) < s$ on $(1-\delta, 1)$.

Now we show that there is at least one solution to $\Phi(s) = s$ in (0,1) when $\mu > 1$. For this, define $f(s) = \Phi(s) - s$ for $s \in \mathbb{R}$. From before, we know $f(0) = \Phi(0) - 0 = \varphi_2(0) > 0$ and moreover, that for $s \in (1-\delta, 1)$, $f(s) = \Phi(s) - s < 0$. Fix s in $(1-\delta, 1)$ and call it s_2. By the Intermediate Value Theorem, there exists $s_4 \in (0, s_2) \subseteq (0,1)$ such that $f(s_4) = 0$, i.e., $\Phi(s_4) = s_4$. Thus, there is at least one solution in $\Phi(s) = s$ in $(0,1)$, when $\mu > 1$.

Finally, we show uniqueness of $s \in (0,1)$ such that $\Phi(s) = s$. Note that if s_4 and s_5 are solutions to $f(s) = 0$, where without loss of generality $0 < s_4 < s_5 < 1$, then by the Mean Value Theorem there exists $s_6 \in (s_4, s_5)$ and $s_7 \in (s_5, 1)$ such that

$$\Phi'(s_6) = \frac{\Phi(s_5) - \Phi(s_4)}{s_5 - s_4} = \frac{s_5 - s_4}{s_5 - s_4} = 1.$$
$$\Phi'(s_7) = \frac{\Phi(1) - \Phi(s_5)}{1 - s_5} = \frac{1 - s_5}{1 - s_5} = 1.$$

Since $\Phi''(s) > 0$ on (0,1), $\Phi'(s)$ is strictly increasing on $(0,1)$ and the above is not possible. Thus, there is exactly one solution to $\Phi(s) = s$ on $(0,1)$. We denote this solution by σ.

In summary, we have shown that $\Phi(s) = s$ has a unique solution given by $s = 1$ in $[0,1]$ if $\mu \leq 1$ and exactly two solutions $s = \sigma > 0$ and $s = 1$ in $[0,1]$ if $\mu > 1$. In particular, we have just proved that if $\mu \leq 1$, then $\Phi(\varepsilon_2) = \varepsilon_2$ and $\varepsilon_2 = 1$, which means that the probability $\hat{\theta}_2$ of percolation of the root is zero if $\mu \leq 1$. On the other hand, if $\mu > 1$, then either $\varepsilon_2 = \sigma$, for the unique $\sigma \in (0,1)$ that satisfies $\varphi_2(\sigma) = \sigma$ or $\varepsilon_2 = 1$.

In order to see that $\varepsilon_2 = \sigma$, let's consider the probability t_n that no vertices in the nth generation are connected to the root by an open path. So

$$t_n = \mathbb{P}(Z_{2,n} = 0), \qquad n \in \mathbb{N}.$$

Recall A_2 is the event that $\{Z_{2,n} = 0 \text{ for some } n \in \mathbb{N}\}$. So, $A_2 = \bigcup_{n\in\mathbb{N}} \{Z_{2,n} = 0\}$, and therefore $\mathbb{P}(A_2) = \mathbb{P}(\bigcup_{n\in\mathbb{N}} \{Z_{2,n} = 0\})$. Note, $\{Z_{2,n} = 0\} \subseteq \{Z_{2,n+1} = 0\}$. Thus, $\lim_{n\to\infty}\{Z_{2,n} = 0\} = A_2$ [7]. Therefore, by [7, Proposition 1.13] $t_n \uparrow \varepsilon_2$ as $n \to \infty$. Here, the sample space is $\Omega = \{0,1\}^\infty$ and $\mathscr{F}$ the σ-field generated by finite dimensional cylinders of Ω gives us the measurable space $(\Omega, \mathscr{F})$ [2]. Note that for $n \in \mathbb{N}$

$$t_{n+1} = \mathbb{P}(Z_{2,n+1} = 0) = \mathbb{P}(Z_{2,1} = 0) + \sum_{k=1}^{2} \mathbb{P}(Z_{2,n+1} = 0, Z_{2,1} = k).$$

We can write out the first term $\mathbb{P}(Z_{2,1} = 0)$, which is $\varphi_2(0)$, and look at the case where $k \geq 1$ separately. By definition of conditional probabilities,

$$\begin{aligned} t_{n+1} =& \varphi_2(0) + \sum_{k=1}^{2} \mathbb{P}(Z_{2,1} = k)\mathbb{P}(Z_{2,n+1} = 0 | Z_{2,1} = k). \\ =& \varphi_2(0) + \mathbb{P}(Z_{2,1} = 1)\mathbb{P}(Z_{2,n+1} = 0 | Z_{2,1} = 1) + \mathbb{P}(Z_{2,1} = 2)\mathbb{P}(Z_{2,n+1} = 0 | Z_{2,1} = 2). \end{aligned}$$

In the case where $k = 1$, there is only one vertex, either V_1 or V_2 (See Figure 1.3) that is connected to the root with an open edge while the other vertex is connected to the root by a close edge. Since the child that is connected to the origin by an open edge e can be thought as a root of binary tree when e is deleted, the probability of $\{Z_{2,n+1} = 0 | Z_{2,1} = 1\}$ is the same as the probability of $\{Z_{2,n} = 0 | Z_{2,0} = 1\}$. Thus, $\mathbb{P}(Z_{2,n+1} = 0 | Z_{2,1} = 1) = \mathbb{P}(Z_{2,n} = 0 | Z_{2,0} = 1) = \mathbb{P}(Z_{2,n} = 0) = t_n$. For $k = 2$, note that each vertex V_1 and V_2 (See Figure 2.2) of the first generation can be thought as a root of a binary tree where these two binary trees have no vertices or edges in common. By deleting the edges between V_1 and $\emptyset$ and V_2 and $\emptyset$, the event $\{Z_{2,n} = 0\}$ for some $n \in \mathbb{N}$ on the subgraph where V_1 is the root or V_2 is the root of their own binary tree is t_2. Hence, $\{Z_{2,n} = 0\}$ for these two subtrees are independent of one another since they are disjoint subtrees, which implies that $\mathbb{P}(Z_{2,n+1} = 0 | Z_{2,1} = 2) = [\mathbb{P}(Z_{2,n} = 0 | Z_{2,0} = 1)]^2 = t_n^2$. Thus,

$$t_{n+1} = \varphi_2(0) + \sum_{k=1}^{2} \varphi_2(k) t_n^k = \Phi(t_n), \qquad n \in \mathbb{N}.$$

Recall that $\Phi(s)$ is strictly increasing on $(0,1)$. Thus, $t_1 = \varphi_2(0) = \Phi(0) < \Phi(\sigma) = \sigma$. So $t_1 < \sigma$. Applying Φ to this inequality and again using the strictly increasing nature of Φ yields $t_2 = \Phi(t_1) < \Phi(\sigma) = \sigma$. The pattern continues and $t_n < \sigma$ for all $n \in \mathbb{N}$. Note that $t_{n+1} = \mathbb{P}(Z_{2,n+1} = 0) \geq \mathbb{P}(Z_{2,n} = 0) = t_n$, so $\{t_n\}$ is a non-decreasing sequence bounded by σ. Thus $\lim_{n\to\infty} t_n \leq \sigma$. Recall that we already knew $t_n \uparrow \varepsilon_2$. Therefore, $\varepsilon_2 \leq \sigma$. Thus, $\varepsilon_2 = \lim_{n\to\infty} t_n \leq \sigma$. But we have already argued that $\Phi(\varepsilon_2) = \varepsilon_2$ and $\Phi(s) \neq s$ for all $0 \leq s < \sigma$, which implies, $\varepsilon_2 \geq \sigma$. Because we have both $\varepsilon_2 \leq \sigma$ and $\varepsilon_2 \geq \sigma$, it must be the case that $\varepsilon_2 = \sigma$.

Consequently, it follows that if $\mu > 1$, then the probability ε_2 of extinction is the unique solution in the interval $[0,1)$ of the equation

$$s = \Phi(s).$$

We can unify the $\mu \leq 1$ and $\mu > 1$ cases by noting that the extinction probability is the smallest nonnegative solution of the equation $s = \Phi(s)$.

Now we use our result to compute ε_2, which is used to find $\hat{\theta}_2$ and $p_{c,2}$. From the earlier discussion if $\mu \leq 1$, then $\varepsilon_2 = 1$. We have $\mu = \Phi'(1) = \varphi_2(1) + 2\varphi_2(2) = 2p(1-p) + 2p^2 = 2p((1-p)+p) = 2p$. Furthermore, $2p \leq 1$ if and only if $p \leq \frac{1}{2}$. Thus $\varepsilon_2 = 1$ when $p \leq \frac{1}{2}$. Equivalently, $\hat{\theta}_2 = 0$ when $p \leq \frac{1}{2}$. This implies $p_{c,2} \geq \frac{1}{2}$, which we already verified. However, $\mu > 1$ when $p > \frac{1}{2}$ and so $0 < \varepsilon_2 < 1$ when $p > \frac{1}{2}$. Therefore, $0 < \hat{\theta}_2 < 1$ when $\frac{1}{2} < p < 1$. So, $p_{c,2} \leq \frac{1}{2}$, as desired.

2.3.2 Inhomogeneous Percolation

The argument used in the homogeneous case can be used in the inhomogeneous case as well.

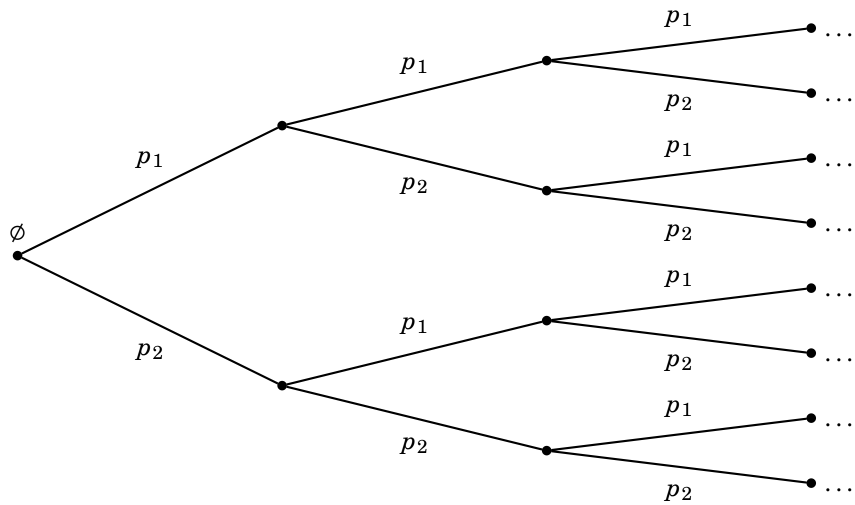

Figure 2.3: Inhomogeneous Binary Tree.

In particular, we generalize by considering $\vec{p} = (p_1, p_2)$, where $p_1 = p_2$ is not required. Figure 2.3 depicts the graph mentioned with edge probability vector $\vec{p} = (p_1, p_2)$. As before, if $p_1 = p_2 = 0$, then there will not be percolation. Also, if $p_1 = 1$ or $p_2 = 1$, we have automatic percolation. If $p_1 = 0$, then p_2 must be equal to 1 for there to be percolation since all edges associated with probability p_2 must be open in order for the root to percolate, and this probability is $\lim_{n\to\infty} p_2^n = 0$ when $0 \le p_2 < 1$. Likewise, if $p_2 = 0$, p_1 must be equal to 1 to have percolation.

We will next determine for which $0 < p_1, p_2 < 1$, $\hat{\theta}_2(\vec{p}) > 0$. To make the next calculation a little more readable, we let $\hat{\theta}_{2,\vec{p}} = \hat{\theta}_2(\vec{p})$, where $\vec{p} = (p_1, p_2)$ so that we may eliminate some parentheses. Henceforth, we assume $0 < p_1 < 1$ and $0 < p_2 < 1$ to avoid the trivial cases treated in previous paragraph. Similarly to the homogeneous case

$$1 - \hat{\theta}_{2,\vec{p}} = (1 - p_1\hat{\theta}_{2,\vec{p}})(1 - p_2\hat{\theta}_{2,\vec{p}}) \tag{2.8}$$

$$= 1 - (p_1 + p_2)\hat{\theta}_{2,\vec{p}} + p_1 p_2 \hat{\theta}_{2,\vec{p}}^2 \tag{2.9}$$

Again, we move all terms to one side, simplify, and factor out a $\hat{\theta}_{2,\vec{p}}$ and we get,

$$\hat{\theta}_{2,\vec{p}}(p_1 p_2 \hat{\theta}_{2,\vec{p}} - (p_1 + p_2) + 1) = 0.$$

Once again, we assume $\hat{\theta}_{2,\vec{p}} > 0$. Then the second factor in the previous line must be zero. Since $p_1, p_2 > 0$, this implies that $\hat{\theta}_{2,\vec{p}} = \frac{p_1+p_2-1}{p_1 p_2}$. Thus, when $\hat{\theta}_{2,\vec{p}} > 0$, $p_1 + p_2 > 1$. Therefore, $\hat{\theta}_{2,\vec{p}} = 0$ for $p_1 + p_2 \leq 1$ and $0 < p_1, p_2 < 1$. The shaded gray area of the coordinate graph in Figure 2.4 shows where $\hat{\theta}_{2,\vec{p}}$ can be positive. Furthermore, the line segments from (0,1) to (1,1) and (1,0) to (1,1), which includes (0,1) and (1,0), are guaranteed to have percolation of the root, but percolation of the root occurs with only probability 0 along the rest of the dashed diagonal segment, i.e., the probability of percolation in is zero on the dashed diagonal line.

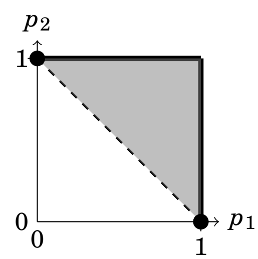

Figure 2.4: The shaded region contains all $\vec{p}$ such that $\hat{\theta}_{2,\vec{p}} > 0$.

As before, we let $\varphi_2(j)$ be the probability of j children, $j = 0, 1, 2$. Then $\varphi_2(0) = (1-p_1)(1-p_2)$, $\varphi_2(1) = p_1(1-p_2) + (1-p_1)p_2$, and $\varphi_2(2) = p_1 p_2$. Hence the generating function $\Phi(s)$ is given by

$$\Phi(s) = \sum_{j=0}^{2} \varphi_2(j)s^j = (1-p_1)(1-p_2) + (p_1(1-p_2) + (1-p_1)p_2)s + p_1 p_2 s^2, \qquad s \in \mathbb{R}.$$

Upon reviewing the proof in the homogeneous case that the probability ε_2 that the root does not percolate is the minimal fixed point of $\Phi(s)$ in $[0,1]$, we see that the only properties used were $\varphi_2(0) > 0$ and $\varphi_2(2) > 0$. Note that $\varphi_2(0) > 0$ and $\varphi_2(2) > 0$ since $0 < p_1, p_2 < 1$. Therefore, $\varepsilon_2 = 1 - \hat{\theta}_{2,\vec{p}}$ satisfies $\varepsilon_2 = \Phi(\varepsilon_2)$. Thus, $\varepsilon_2 = 1$ and $\hat{\theta}_{2,\vec{p}} = 0$, if $\Phi'(1) \leq 1$. Also, $0 < \varepsilon_2 < 1$ and $0 < \hat{\theta}_{2,\vec{p}}$, if $\Phi'(1) > 1$. Just like in Section 2.3.1, this can be used to give an alternative derivation of (2.8) and (2.9). Likewise, $\mu = \Phi'(1)$, and

$$\Phi'(1) = [(1-p_1)p_2 + (1-p_2)p_1] + 2p_1p_2 = [p_1 + p_2 - 2p_1p_2] + 2p_1p_2 = p_1 + p_2.$$

So, $\hat{\theta}_{2,\vec{p}} = 0$ when $p_1, p_2 < 1$ and $p_1 + p_2 \leq 1$ and $\hat{\theta}_{2,\vec{p}} > 0$ when $p_1 = 1, p_2 = 1$, or $p_1 + p_2 > 1$. Since the critical surface is defined as the boundary set of the parameters for which $\hat{\theta}_{2,\vec{p}}$ is positive, we see that the critical surface is the set of ordered pairs such that $p_1 + p_2 = 1$ and that $\hat{\theta}_{2,\vec{p}} > 0$ if and only if $p_1 + p_2 > 1$, p_1 or $p_2 = 1$. We refer to the boundary separating the shaded and unshaded region as the *critical surface*. In particular, the dashed line in Figure 2.4 is the critical surface and the root percolates with positive probability on the entire shaded region.

2.4 Inhomogenous Percolation on the Trinary Tree

For the trinary case, having gained some intuition, we can go straight to the inhomogeneous bond case, letting $\hat{\theta}_{3,\vec{p}} = \hat{\theta}_3(\vec{p})$, where $\vec{p} = (p_1, p_2, p_3)$ (See Figure 2.5).

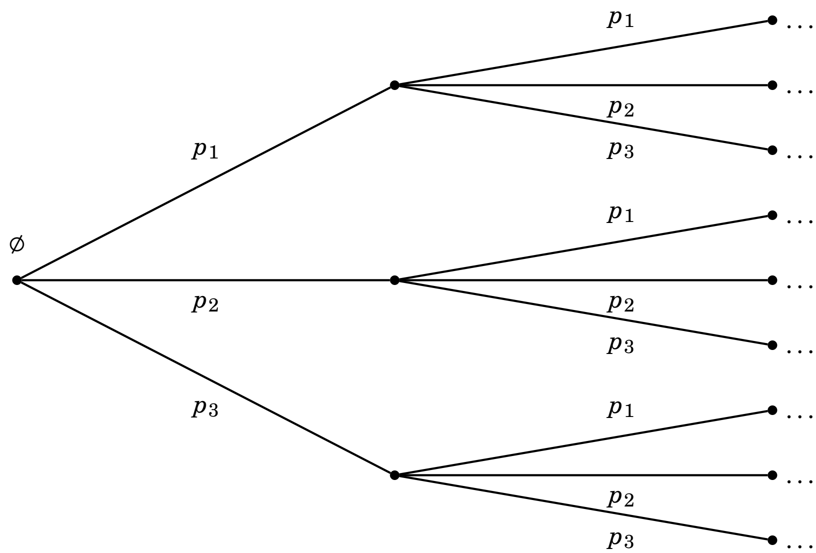

Figure 2.5: Inhomogeneous Trinary Tree.

Using the same reasoning as in the binary case,

$$\begin{aligned} 1-\hat{\theta}_{3,\vec{p}} =& (1-p_1\hat{\theta}_{3,\vec{p}})(1-p_2\hat{\theta}_{3,\vec{p}})(1-p_3\hat{\theta}_{3,\vec{p}}) \\ =& 1-(p_1+p_2+p_3)\hat{\theta}_{3,\vec{p}} + (p_1p_2+p_1p_3+p_2p_3)\hat{\theta}^2_{3,\vec{p}} - p_1p_2p_3\hat{\theta}^3_{3,\vec{p}}. \end{aligned} \tag{2.10}$$

After subtracting 1 from both sides, moving all terms to one side and factoring out a $\hat{\theta}_{3,\vec{p}_3}$ we get,

$$\hat{\theta}_{3,\vec{p}}(p_1p_2p_3\hat{\theta}^2_{3,\vec{p}} - (p_1p_2+p_1p_3+p_2p_3)\hat{\theta}_{3,\vec{p}} + (p_1+p_2+p_3-1)) = 0. \tag{2.11}$$

If $p_i = 0$ for some $i = 1,2$, or 3, percolation in this scenario is equivalent to the binary case. Equation (2.10) becomes (2.8) in this case since the root is now in a binary tree subgraph. For example, we get $\hat{\theta}_{3,\vec{p}} = \frac{p_1+p_2-1}{p_1p_2}$ if $p_3 = 0$, $p_1, p_2 > 0$, and $p_1+p_2-1 > 0$ which is consistent with our earlier results. If two parameters were equal to zero, we would have at best a graph where each vertex has two edges except for the root and our nonzero parameter would have to be equal to 1 for percolation to occur. Of course, if $p_1 = p_2 = p_3 = 0$, then $\hat{\theta}_{3,\vec{p}} = 0$.

Thus, assuming $p_1, p_2, p_3 > 0$ and $\hat{\theta}_{3,\vec{p}} > 0$, we use the quadratic equation for the quadratic factor in (2.11) to obtain that,

$$\hat{\theta}_{3,\vec{p}} = \frac{p_1p_2 + p_1p_3 + p_2p_3 \pm \sqrt{(p_1p_2 + p_1p_3 + p_2p_3)^2 - 4p_1p_2p_3(p_1 + p_2 + p_3 - 1)}}{2p_1p_2p_3}. \tag{2.12}$$

We must argue that the radicand is nonnegative for $p_1, p_2, p_3 > 0$, determine whether to add or subtract the radical, and determine when the expression on the right side of (2.12) is actually positive.

We begin by assuming that $p_1, p_2, p_3 > 0$ and the radicand in (2.12) is nonnegative. Under this assumption, we verify that subtracting the radical is the correct choice. As motivation for this, note that when subtracting the radical and letting p_1, p_2, and p_3 tend to 1, the expression on the right side of (2.12) is 1. On the other hand, when adding the radical the limiting expression is 2. More generally, note that, for $p_1, p_2, p_3 > 0$,

$$\begin{aligned}
&\frac{p_1p_2 + p_1p_3 + p_2p_3 + \sqrt{(p_1p_2 + p_1p_3 + p_2p_3)^2 - 4p_1p_2p_3(p_1 + p_2 + p_3 - 1)}}{2p_1p_2p_3} \\
&= \frac{1}{2p_3} + \frac{1}{2p_2} + \frac{1}{2p_1} + \frac{\sqrt{(p_1p_2 + p_1p_3 + p_2p_3)^2 - 4p_1p_2p_3(p_1 + p_2 + p_3 - 1)}}{2p_1p_2p_3} \\
&\geq \frac{1}{2p_3} + \frac{1}{2p_2} + \frac{1}{2p_1} \geq \frac{3}{2},
\end{aligned}$$

yet $0 < \hat{\theta}_{3,\vec{p}} \leq 1$. Therefore, addition cannot be correct for any $p_1, p_2, p_3 > 0$. So if the equality in (2.12) holds, it holds with the radicand subtracted.

Next we determine for which $p_1, p_2, p_3 > 0$, the expression on the right of (2.12) with the radical subtracted is positive. We have.

$$\frac{p_1p_2 + p_1p_3 + p_2p_3 - \sqrt{(p_1p_2 + p_1p_3 + p_2p_3)^2 - 4p_1p_2p_3(p_1 + p_2 + p_3 - 1)}}{2p_1p_2p_3} > 0, \tag{2.13}$$

if and only if

$$p_1p_2 + p_1p_3 + p_2p_3 - \sqrt{(p_1p_2 + p_1p_3 + p_2p_3)^2 - 4p_1p_2p_3(p_1 + p_2 + p_3 - 1)} > 0.$$

We then move the radical to the other side of the inequality and square both sides to obtain that (2.13) holds if and only if

$$(p_1p_2 + p_1p_3 + p_2p_3)^2 > (p_1p_2 + p_1p_3 + p_2p_3)^2 - 4p_1p_2p_3(p_1 + p_2 + p_3 - 1).$$

Once we subtract $(p_1p_2 + p_1p_3 + p_2p_3)^2$ from both sides and multiply by a negative one, this results in that (2.13) holds if and only if $0 < 4p_1p_2p_3(p_1 + p_2 + p_3 - 1)$. Thus we get that $p_1 + p_2 + p_3 - 1 > 0$ is necessary and sufficient for (2.13) to hold.

Note that when $0 < p_1 + p_2 + p_3 \le 1$,

$$p_1p_2 + p_1p_3 + p_2p_3 - \sqrt{(p_1p_2 + p_1p_3 + p_2p_3)^2 - 4p_1p_2p_3(p_1 + p_2 + p_3 - 1)} \le 0.$$

However, if $\hat{\theta}_{3,\vec{p}} > 0$, then (2.12) holds. Thus, $\hat{\theta}_{3,\vec{p}} = 0$ when $p_1, p_2, p_3 > 0$ and $p_1 + p_2 + p_3 \le 1$.

Finally, we verify that the radicand in (2.12) is strictly positive when $1 < p_1 + p_2 + p_3 \le 3$. After simplifying the radicand in (2.12), we get

$$\begin{aligned}
&p_1^2p_2^2 + p_1^2p_3^2 + p_2^2p_3^2 + 2p_1^2p_2p_3 + 2p_1p_2^2p_3 + 2p_1p_2p_3^2 - 4p_1^2p_2p_3 - 4p_1p_2^2p_3 - 4p_1p_2p_3^2 + 4p_1p_2p_3 \\
&= p_1^2p_2^2 + p_1^2p_3^2 + p_2^2p_3^2 - 2p_1^2p_2p_3 - 2p_1p_2^2p_3 - 2p_1p_2p_3^2 + 4p_1p_2p_3 \\
&= p_1^2p_2^2 + p_1^2p_3^2 + p_2^2p_3^2 + 2p_1p_2p_3(2 - (p_1 + p_2 + p_3)).
\end{aligned}$$

The last line is clearly positive when $1 < p_1 + p_2 + p_3 \le 2$. The difficult case is to show the radicand is positive is when $2 < p_1 + p_2 + p_3 < 3$. Without loss of generality, suppose $0 < p_1 \le p_2 \le p_3$. Then $p_1^2p_2^2 \le p_1^2p_3^2 \le p_2^2p_3^2$. This gives,

$$\begin{aligned}
&p_1^2p_2^2 + p_1^2p_3^2 + p_2^2p_3^2 + 2p_1p_2p_3(2 - (p_1 + p_2 + p_3)) \\
&\ge 3p_1^2p_2^2 + 2p_1p_2p_3(2 - (p_1 + p_2 + p_3)) \\
&= p_1p_2(3p_1p_2 - 2p_1p_3 + 2p_3 - 2p_2p_3 + 2p_3 - 2p_3^2) \\
&= p_1p_2[p_1(3p_2 - 2p_3) + 2p_3(1 - p_2) + 2p_3(1 - p_3)] \\
&= p_1p_2[p_1(3p_2 - 2p_3) + 2p_3(2 - p_2 - p_3)].
\end{aligned}$$

The factor $2 - p_2 - p_3$ is nonnegative because $p_2, p_3 \le 1$ and the factor $3p_2 - 2p_3$ is nonnegative if $p_2 \ge \frac{2}{3}p_3$. Note, that $\frac{1}{2} < p_2$, otherwise $p_1, p_2 \le \frac{1}{2}$ and $p_1 + p_2 + p_3 \le 2$, which contradicts $2 < p_1 + p_2 + p_3 \le 3$. When $\frac{1}{2} < p_2$ and $p_2 < \frac{2}{3}p_3$, then,

$$0 > 3p_2 - 2p_3 > 3\left(\frac{1}{2}\right) - 2p_3 = \frac{3}{2} - 2p_3 \ge -\frac{1}{2}.$$

Therefore,

$$\begin{aligned}
&2p_3(2-p_2-p_3)+p_1(3p_2-2p_3)\\
&> 2p_3(2-p_2-p_3)+p_1(-1/2)\\
&\geq 2p_3(2-p_2-p_3)+p_3(-1/2)\\
&= p_3(4-2p_2-2p_3-1/2)\\
&> p_3(4-2(2/3)-2-1/2)\\
&= p_3(1/6)>0.
\end{aligned}$$

In conclusion, when $p_1,p_2,p_3>0$ and $p_1+p_2+p_3>1$, the radicand in (2.12) is strictly positive. So we have verified that if $p_1,p_2,p_3>0$, $p_1+p_2+p_3>1$, and $\hat{\theta}_{3,\vec{p}}>0$, then

$$\hat{\theta}_{3,\vec{p}}=\frac{p_1p_2+p_1p_3+p_2p_3-\sqrt{(p_1p_2+p_1p_3+p_2p_3)^2-4p_1p_2p_3(p_1+p_2+p_3-1)}}{2p_1p_2p_3}. \tag{2.14}$$

Further, $\hat{\theta}_{3,\vec{p}}=0$ when $0\leq p_1+p_2+p_3\leq 1$.

In order to show that $p_1+p_2+p_3=1$ is the critical surface, we must show that $\hat{\theta}_{3,\vec{p}}>0$ when $p_1+p_2+p_3>1$. As before, we use the generating function, but for the inhomogeneous trinary case. In particular, let $\varphi_3(k)$ be the probability that k edges are open between a single vertex and its children for the trinary tree, for $0\leq k\leq 3$. Also, for $s\in\mathbb{R}$, $\Phi(s)=\sum_{k=0}^{3}\varphi_3(k)s^k$. Observe that $\varphi_3(0)=(1-p_1)(1-p_2)(1-p_3)$, which is positive if p_1,p_2,p_3 are all less than 1. Also, if $0<p_1,p_2,p_3<1$,

$$\varphi_3(0)+\varphi_3(1)=1-\varphi_3(2)-\varphi_3(3)=1-[p_1p_2(1-p_3)+p_1(1-p_2)p_3+(1-p_1)p_2p_3]-p_1p_2p_3<1,$$

which is the condition needed to conclude that ε_3, i.e., the probability of $A_3=\bigcup_{n=0}^{\infty}\{Z_{3,n}=0\}$, where $Z_{3,n}$ is the total number of all vertices in the trinary tree of distance n from the root that are connected to the root by an open path for $n\in\mathbb{Z}_+$, is the minimal fixed point of $\Phi(s)$ in $[0,1]$. Since $\varepsilon_3=1-\hat{\theta}_{3,\vec{p}}$, $\hat{\theta}_{3,\vec{p}}>0$ if and only if $\varepsilon_3<1$, if and only if $\Phi'(1) > 1$. We now calculate $\Phi'(1)$ as follows:

$$\begin{aligned}
\Phi'(1) =& \varphi_3(1) + 2\varphi_3(2)) + 3\varphi_3(3) \\
=& p_1(1-p_2)(1-p_3) + p_2(1-p_1)(1-p_3) + p_3(1-p_1)(1-p_2) \\
& + 2[(1-p_1)p_2p_3 + (1-p_2)p_1p_3 + (1-p_3)p_1p_2] + 3[p_1p_2p_3] \\
=& p_1 - p_1p_2 - p_1p_3 + p_1p_2p_3 + (1-p_1)(p_2+p_3-2p_2p_3) \\
& + 2p_2p_3 + 2p_1p_3 + 2p_1p_2 - 6p_1p_2p_3 + 3p_1p_2p_3 \\
=& p_1 - p_1p_2 - p_1p_3 + p_1p_2p_3 + p_2 + p_3 - 2p_2p_3 - p_1p_2 - p_1p_3 + 2p_1p_2p_3 \\
& + 2p_2p_3 + 2p_1p_3 + 2p_1p_2 - 3p_1p_2p_3 \\
=& p_1 + p_2 + p_3.
\end{aligned}$$

Thus, $0 < \varepsilon_3 < 1$ when $p_1 + p_2 + p_3 > 1$ and so $0 < \hat{\theta}_{3,\vec{p}} < 1$ when $p_1 + p_2 + p_3 > 1$. Therefore, the critical surface is the set of points in $[0,1]^3$ that satisfy the equation $p_1 + p_2 + p_3 = 1$.

2.5 Inhomogeneous Percolation on r-ary Trees

Here, we consider the general case, where we have an r-ary tree with inhomogeneous bond percolation as indicated in the graph in Figure 2.6. As we have done before, $\hat{\theta}_{r,\vec{p}}$ is the probability that the root percolates, where $\vec{p} = (p_1, p_2, ..., p_r)$, $r \geq 2$, and r is finite. We wish to determine the critical surface for general r.

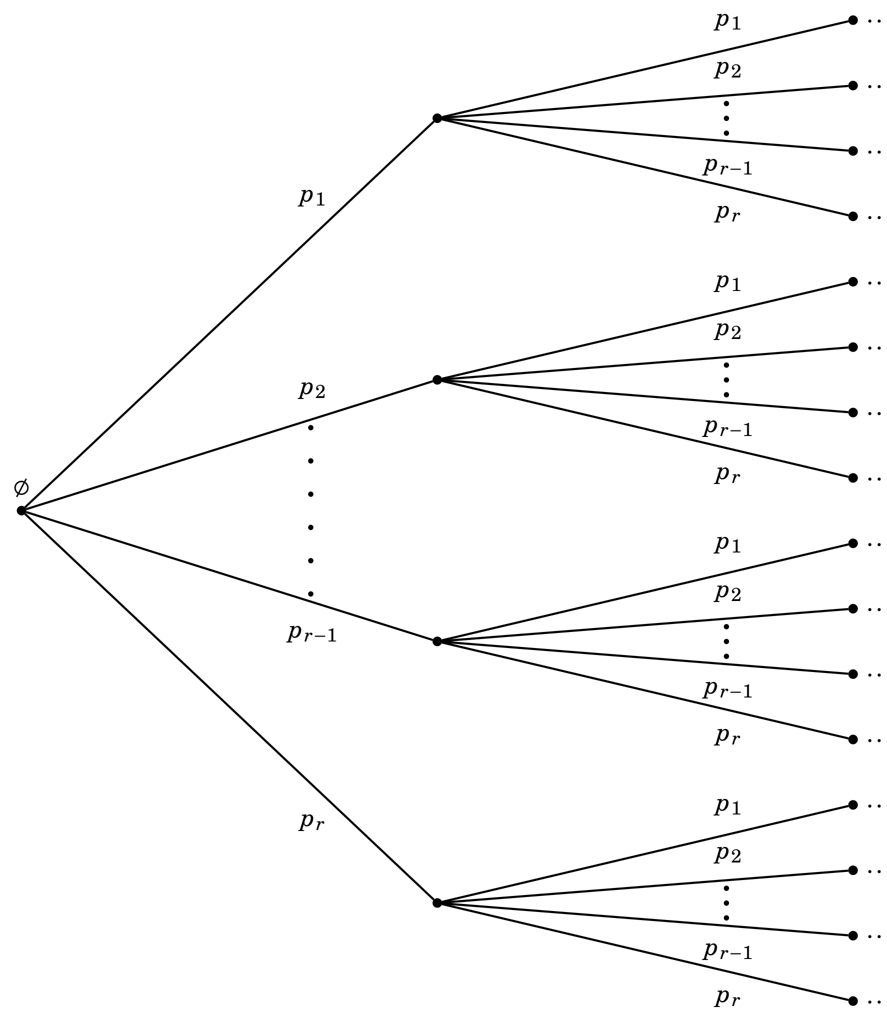

Figure 2.6: Inhomogeneous r-ary Tree.

A key to the analysis in the cases $r = 2$ and $r = 3$ was to study fixed points of the generating function. For this, for $0 \le k \le r$, let C_k be the event that exactly k of the r edges leading to the children of a single vertex are open. Set $\varphi_r(k) = \mathbb{P}(C_k)$ for $0 \le k \le r$. Define the generating function $\Phi_r(s)$, for $s \in \mathbb{R}$, as,

$$\Phi_r(s) = \sum_{k=0}^{r} \varphi_r(k) s^k. \tag{2.15}$$

Let μ denote the expected number of open edges leading to the children of a single vertex. Then,

$$\mu = \sum_{k=0}^{r} k\mathbb{P}(C_k) = \sum_{k=0}^{r} k\varphi_r(k) = \Phi'(1).$$

Theorem 1. *If $p_i < 1$ for all i and $p_i > 0$ for at least two i, then $\varepsilon_r = 1 - \hat{\theta}_{r,\vec{p}}$ (which is the probability that the* *root does not percolate) is the minimal positive fixed point of* Φ *in [0,1]. Furthermore,* $0 < \varepsilon_r < 1$ *if and only if* $\mu > 1$*, and* $\varepsilon_r = 1$ *if and only if* $\mu \leq 1$.

In Section 2.3.1, we proved this theorem for $r = 2$. In Section 2.4, we commented on why that proof extends to $r = 3$. The key facts needed to extend the proof are $\varphi_3(0) > 0$ and $\varphi_3(0) + \varphi_3(1) < 1$. To extend the proof to $r \geq 4$, we simply need to verify that $\varphi_r(0) > 0$ and $\varphi_r(0) + \varphi_r(1) < 1$.

Remark 1. $\varphi_r(0) = \prod_{i=1}^{r}(1 - p_i) > 0$ *if and only if* $p_i < 1$ *for all* i.

Remark 2. $\varphi_r(0) + \varphi_r(1) < 1$ *if and only if* $\sum_{k=2}^{r} \varphi_r(k) > 0$ *if and only if* $p_i > 0$ *for at least two* i*. To verify the second equivalence, suppose* p_j *and* p_k *positive for* $j \neq k$*. Then since* $p_i < 1$ *for all* i*,* $\sum_{k=2}^{r} \varphi_r(k) \geq \varphi_r(2) \geq p_j p_k \prod_{i \neq j,k}(1 - p_i) > 0$.

Corollary 1. *If* $p_i < 1$ *for all* i *and* $p_i > 0$ *for at least two* i*, then* $\hat{\theta}_{r,\vec{p}} = 0$ *if* $\mu \leq 1$ *and* $\hat{\theta}_{r,\vec{p}} > 0$ *if* $\mu > 1$.

Remark 3. *There are two special cases not included in Corollary 1, that are rather trivial. The first is* $p_i = 1$ *for some* i*. Then* $\hat{\theta}_{r,\vec{p}} = 1$*. The second is if there exists* j *such that* $p_j < 1$ *and* $p_i = 0$ *for all* $i \neq j$ *(This includes* $p_i = 0$ *for all* i*). Then,* $\hat{\theta}_{r,\vec{p}} = 0$*. These two cases are included in Theorem 2.*

We obtain the following result by combining Theorem 1, Remark 3 and the fact that $\mu = p_1 + ... + p_r$ (proved in the proof of Theorem 2 below).

Theorem 2. *If* $p_1 + p_2 + ... + p_r > 1$ *or* $p_i = 1$ *for some* i*, then* $\hat{\theta}_{r,\vec{p}} > 0$*. If* $p_1 + p_2 + ... + p_r \leq 1$*, and* $p_i < 1$ *for all* i*, then* $\hat{\theta}_{r,\vec{p}} = 0$.

Proof. Let Y be a random variable that equals the number of open edges leading to the children of a the root. The expected value of Y is $\mathbb{E}[Y] = \sum_{i=1}^{r} i\varphi_r(i) = \mu$. Note that $Y = \mathbb{1}_{B_1} + \mathbb{1}_{B_2} + ... + \mathbb{1}_{B_r}$, where $B_i =$ { the edge between the i^{th} child of the root and the root is open } and $\mathbb{1}_{B_i}$ is the indicator function of B_i,

$$\mathbb{1}_{B_i} = \begin{cases} 1, & \text{if } B_i \text{ occurs.} \\ 0, & \text{otherwise.} \end{cases}$$

By computing the expected value of each $\mathbb{1}_{B_i}$, we find that $\mathbb{E}\left[\mathbb{1}_{B_i}\right] = 0 \cdot \mathbb{P}(\mathbb{1}_{B_i^C}) + 1 \cdot \mathbb{P}(\mathbb{1}_{B_i}) = p_i$. By linearity, $\mathbb{E}[Y] = \mathbb{E}\left[\mathbb{1}_{B_1} + \mathbb{1}_{B_2} + ... + \mathbb{1}_{B_r}\right] = \mathbb{E}\left[\mathbb{1}_{B_1}\right] + \mathbb{E}\left[\mathbb{1}_{B_2}\right] + ... + \mathbb{E}\left[\mathbb{1}_{B_r}\right] = p_1 + p_2 + ... + p_r$. So, $\mu = p_1 + p_2 + \cdots + p_r$. Then the result follows from Corollary 1 and Remark 3. □

As an immediate consequence of Theorem 2, we obtain the following corollary.

Corollary 2. *The critical surface of* $\hat{\theta}_{r,\vec{p}}$ *is* $p_1 + ... + p_r = 1$.

In addition to knowing the critical surface, we would also like to obtain a formula for $\hat{\theta}_{r,\vec{p}}$, when $\hat{\theta}_{r,\vec{p}} > 0$. However, it is difficult to obtain a closed form expression of $\hat{\theta}_{r,\vec{p}}$ because of the increased complexity of the equality shown in Theorem 3 for general r. However, we can prove the following theorem, which identifies $\hat{\theta}_{r,\vec{p}}$ as a root of a polynomial of r^{th} degree.

Theorem 3. *Let* $\hat{\theta}_r = \hat{\theta}_{r,\vec{p}}$. *Then the percolation probability satisfies,*

$$1-\hat{\theta}_r = 1-\left(\sum_{i_1=1}^{r} p_{i_1}\right)\hat{\theta}_r + \left(\sum_{1\le i_1<i_2\le r} p_{i_1}p_{i_2}\right)\hat{\theta}_r^2 - \left(\sum_{1\le i_1<i_2<i_3\le r} p_{i_1}p_{i_2}p_{i_3}\right)\hat{\theta}_r^3 + ... + (-1)^r\left(\prod_{i_r=1}^{r} p_{i_r}\right)\hat{\theta}_r^r. \tag{2.16}$$

Proof. Not percolating at the root of an r-ary tree is $1-\hat{\theta}_{r,\vec{p}}$. Removing the edge between $\emptyset$ and its child V_i where associated probability parameter of the edge is p_i for some fixed $i \in \{1,2,...,r\}$, forms a new r-ary tree where the root is V_i. Percolating on V_i is $\hat{\theta}_{r,\vec{p}}$ and having the edge between $\emptyset$ and V_i open is p_i so percolating from $\emptyset$ using this edge is $p_i\hat{\theta}_{r,\vec{p}}$. Thus, not percolating on the r-ary tree on the subgraph with root V_i is $1-p_i\hat{\theta}_{r,\vec{p}}$. Similarly, we can say the same for the rest of the children V_j where $\{1,2,...,r\}$ of $\emptyset$. Since these events are independent of each other, we can say $1-\hat{\theta}_{r,\vec{p}} = (1-p_1\hat{\theta}_{r,\vec{p}})(1-p_2\hat{\theta}_{r,\vec{p}})...(1-p_r\hat{\theta}_{r,\vec{p}})$. Using algebra to expand the product on the left hand side we get,

$$1-\hat{\theta}_{r,\vec{p}} =(1-p_1\hat{\theta}_{r,\vec{p}})(1-p_2\hat{\theta}_{r,\vec{p}})...(1-p_r\hat{\theta}_{r,\vec{p}}). \tag{2.17}$$

Expanding (2.17) results in (2.16). □

Alternate Proof. Recall the definition of the generating function Φ_r, given in (2.15). Generalizing the notation introduced in the cases where $r = 2$ and $r = 3$, we write ε_r for the probability of $\{Z_{r,n} = 0$ for some $n \in \mathbb{N}\}$. Proceeding similarly to how we derived (2.6), we have that ε_r is a fixed point of Φ_r, i.e., $\varepsilon_r = \Phi_r(\varepsilon_r)$ or $1-\hat{\theta}_{r.\vec{p}} = \Phi_r(1-\hat{\theta}_{r,\vec{p}})$. We now use the generating function and prove that $\sum_{i=0}^{r}\varphi_r(i)(1-\hat{\theta}_{r,\vec{p}})^i = \sum_{i=0}^{r}(-1)^i\sum_{j=i}^{r}\binom{j}{i}\varphi_r(j)\hat{\theta}_{r,\vec{p}}^i$. For $r \in \mathbb{N}$, we focus on the left hand side of the last expression, use the Binomial Theorem to expand $(1-\hat{\theta}_{r,\vec{p}})^i$ and then interchange the order of summation to get

$$1-\hat{\theta}_{r,\vec{p}} = \sum_{i=0}^{r}\varphi_r(i)(1-\hat{\theta}_{r,\vec{p}})^i = \sum_{i=0}^{r}\varphi_r(i)\sum_{k=0}^{i}\binom{i}{k}(1)^{i-k}(-\hat{\theta}_{r,\vec{p}})^k = \sum_{k=0}^{r}\sum_{i=k}^{r}\varphi_r(i)\binom{i}{k}(-1)^k\hat{\theta}_{r,\vec{p}}^k.$$

We now change variables by letting $i = j$ first, $k = i$ second and then factoring out $(-1)^i$ hence

$$\begin{aligned}1-\hat{\theta}_{r,\vec{p}} =& \sum_{i=0}^{r}(-1)^i \sum_{j=i}^{r} \binom{j}{i} \varphi_r(j)\hat{\theta}^i_{r,\vec{p}} \\ =& 1-\sum_{j_1=1}^{r}\binom{j_1}{1}\varphi_r(j_1)\hat{\theta}_{r,\vec{p}}+\sum_{j_2=2}^{r}\binom{j_2}{2}\varphi_r(j_2)\hat{\theta}^2_{r,\vec{p}}-\dots+(-1)^r\sum_{j_r=r}^{r}\binom{j_r}{r}\varphi_r(j_r)\hat{\theta}^r_{r,\vec{p}}. \end{aligned} \tag{2.18}$$

Next we show that

$$1-\hat{\theta}_{r,\vec{p}} = 1-\left(\sum_{i_1=1}^{r} p_{i_1}\right)\hat{\theta}_{r,\vec{p}}+\left(\sum_{1\le i_1<i_2\le r} p_{i_1}p_{i_2}\right)\hat{\theta}^2_{r,\vec{p}}-\dots+(-1)^r\left(\prod_{i_r=1}^{r} p_{i_r}\right)\hat{\theta}^r_{r,\vec{p}}. \tag{2.19}$$

In particular, we show that $\sum_{j_m=m}^{r}\binom{j_m}{m}\varphi_r(j_m) = \sum_{1\le i_1<\dots<i_m\le r} p_{i_1}\cdots p_{i_m}$ where $1 \le m \le r$.

In order to achieve this, we let $S_r(j) = \sum_{1\le i_1<i_2<\dots<i_j\le r} p_{i_1}p_{i_2}\dots p_{i_j}$ for $r \in \mathbb{N}$, $j \in \mathbb{Z}_+$ where $1 \le j \le r$. Also, we adopt the convention $S_r(0) = 1$. For example, $S_r(r) = p_1p_2\cdots p_r$. We want to show that $1-\hat{\theta}_{r,\vec{p}} = 1-S_r(1)\hat{\theta}_{r,\vec{p}}+S_r(2)\hat{\theta}^2_{r,\vec{p}}-\dots+(-1)^rS_r(r)\hat{\theta}^r_{r,\vec{p}}$ to prove this (2.19). Our approach will be first to prove that

$$\varphi_r(j) = \sum_{l=j}^{r}(-1)^{l+j}\binom{l}{j}S_r(l), \tag{2.20}$$

for $j = 0,\dots,r$, and once we have obtained (2.20), we will invert this relation.

The proof of (2.20) is by induction.

Base Step: Take $r = 1$. For $j = 0$, $\sum_{l=0}^{1}(-1)^{l+0}\binom{l}{0}S_1(l) = (-1)^{0+0}\binom{0}{0}S_1(0)+(-1)^{1+0}\binom{1}{0}S_1(1) = 1-S_1(1) = 1-p_1 = \varphi_1(0)$. For $j = 1$, $\sum_{l=1}^{1}(-1)^{l+1}\binom{l}{1}S_1(l) = (-1)^{1+1}\binom{1}{1}S_1(1) = p_1 = \varphi_r(1)$.

Induction Hypothesis: Suppose for some $r \in \mathbb{N}$ that $\varphi_r(j) = \sum_{l=j}^{r}(-1)^{l+j}\binom{l}{j}S_r(l)$ for $j = 0,\dots,r$.

Induction Step: We break the proof that the formula holds for $r+1$ into three cases: $j = 0, j = r+1$ and $j \in \{1,\dots,r\}$

For $j = 0$, $\varphi_{r+1}(0) = \prod_{i=1}^{r+1}(1-p_i) = (1-p_{r+1})\prod_{i=1}^{r}(1-p_i) = (1-p_{r+1})\varphi_r(0)$. Thus, by the inductive hypothesis,

$$\begin{aligned}\varphi_{r+1}(0) =& (1-p_{r+1})\varphi_r(0) = (1-p_{r+1})\sum_{l=0}^{r}(-1)^{l+0}\binom{l}{0}S_r(l) = (1-p_{r+1})\sum_{l=0}^{r}(-1)^lS_r(l) \\ =& S_r(0)+\sum_{l=1}^{r}\left[(-1)^lS_r(l)-p_{r+1}(-1)^{l-1}S_r(l-1)\right]-p_{r+1}(-1)^rS_r(r) \\ =& 1+\sum_{l=1}^{r}(-1)^l\left[S_r(l)+p_{r+1}S_r(l-1)\right]+(-1)^{r+1}S_{r+1}(r+1) \end{aligned} \tag{2.21}$$

We would like to show that the bracketed term in (2.21) collapses to $S_{r+1}(l)$. To see this, note that we can divide the l-fold products of probability parameters that are summed in $S_{r+1}(l)$ into two groups: those that do not include a factor of p_{r+1} and those that do. The sum of all l-fold products of probability parameters drawn from $\{p_1, p_2, ..., p_{r+1}\}$ that do not include a factor of p_{r+1} is $S_r(l)$, and the sum of all l-fold products of probability parameters drawn from $\{p_1, p_2, ..., p_{r+1}\}$ where one of the factors is required to be p_{r+1} can be written $p_{r+1}S_r(l-1)$. Therefore, $S_{r+1}(l) = S_r(l) + p_{r+1}S_r(l-1)$, and so

$$\varphi_{r+1}(0) = S_{r+1}(0) + \sum_{l=1}^{r+1}(-1)^l S_{r+1}(l) = \sum_{l=0}^{r+1}(-1)^l S_{r+1}(l) = \sum_{l=0}^{r+1}(-1)^l \binom{l}{0} S_{r+1}(l).$$

Thus, (2.20) holds for $r+1$ and $j=0$.

For $j = r+1$, $\varphi_{r+1}(r+1) = p_1 ... p_{r+1} = p_{r+1}\prod_{i=1}^{r} p_i = p_{r+1}\varphi_r(r)$. Thus, by the inductive hypothesis,

$$\varphi_{r+1}(r+1) = p_{r+1}\varphi_r(r) = p_{r+1}\sum_{l=r}^{r}(-1)^{l+r}\binom{l}{r}S_r(r) = p_{r+1}S_r(r) = S_{r+1}(r+1).$$

For $j \in \{1, ..., r\}$,

$$\begin{aligned}
\varphi_{r+1}(j) =& (1-p_{r+1})\varphi_r(j) + p_{r+1}\varphi_r(j-1) \\
=& (1-p_{r+1})\sum_{l=j}^{r}(-1)^{l+j}\binom{l}{j}S_r(l) + p_{r+1}\sum_{l=j-1}^{r}(-1)^{l+j-1}\binom{l}{j-1}S_r(l) \\
=& \sum_{l=j}^{r}(-1)^{l+j}\binom{l}{j}S_r(l) - \sum_{l=j}^{r}(-1)^{l+j}\binom{l}{j}p_{r+1}S_r(l) + p_{r+1}S_r(j-1) + \sum_{l=j}^{r}(-1)^{l+j-1}\binom{l}{j-1}p_{r+1}S_r(l) \\
=& \sum_{l=j}^{r}(-1)^{l+j}\binom{l}{j}S_r(l) + p_{r+1}S_r(j-1) + \sum_{l=j}^{r}(-1)^{l+j-1}\left[\binom{l}{j} + \binom{l}{j-1}\right]p_{r+1}S_r(l) \\
=& \sum_{l=j}^{r}(-1)^{l+j}\binom{l}{j}S_r(l) + p_{r+1}S_r(j-1) + \sum_{l=j}^{r}(-1)^{l+j-1}\binom{l+1}{j}p_{r+1}S_r(l) \\
=& \sum_{l=j}^{r}(-1)^{l+j}\binom{l}{j}S_r(l) + \sum_{l=j-1}^{r}(-1)^{l+j-1}\binom{l+1}{j}p_{r+1}S_r(l) \\
=& \sum_{l=j}^{r}(-1)^{l+j}\binom{l}{j}S_r(l) + \sum_{l'=j}^{r+1}(-1)^{l'+j}\binom{l'}{j}p_{r+1}S_r(l'-1) \\
=& \sum_{l=j}^{r}(-1)^{l+j}\binom{l}{j}S_r(l) + \sum_{l'=j}^{r}(-1)^{l'+j}\binom{l'}{j}p_{r+1}S_r(l'-1) + (-1)^{r+j+1}\binom{r+1}{j}p_{r+1}S_r(r) \\
=& \sum_{l=j}^{r}(-1)^{l+j}\binom{l}{j}\left[S_r(l) + p_{r+1}S_r(l-1)\right] + (-1)^{r+j+1}\binom{r+1}{j}p_{r+1}S_r(r) \\
=& \sum_{l=j}^{r}(-1)^{l+j}\binom{l}{j}S_{r+1}(l) + (-1)^{r+j+1}\binom{r+1}{j}S_{r+1}(r+1) \\
=& \sum_{l=j}^{r+1}(-1)^{l+j}\binom{l}{j}S_{r+1}(l).
\end{aligned}$$

This completes the induction, so (2.20) holds, and we can continue on with proving (2.19). We now expand (2.20).

$$\begin{array}{rcccccccccccc}
\varphi_r(0) = & S_r(0) & - & S_r(1) & + & S_r(2) & - & S_r(3) & + & S_r(4) & - & S_r(5) & + \ldots + (-1)^{r+0} S_r(r) \\
\varphi_r(1) = & & & S_r(1) & - & 2S_r(2) & + & 3S_r(3) & - & 4S_r(4) & + & 5S_r(5) & + \ldots + (-1)^{r+1} r S_r(r) \\
\varphi_r(2) = & & & & & S_r(2) & - & 3S_r(3) & + & 6S_r(4) & - & 10S_r(5) & + \ldots + (-1)^{r+2} \binom{r}{2} S_r(r) \\
\vdots & & & & & & \ddots & & & & & & \vdots \\
\varphi_r(n) = & & & & & & & \binom{n}{n} S_r(n) & - & \binom{n+1}{n} S_r(n+1) & + & \ldots\ldots\ldots\ldots & + \ldots + (-1)^{r+n} \binom{r}{n} S_r(r) \\
\vdots & & & & & & & & & \ddots & & & \vdots \\
\varphi_r(r-1) = & & & & & & & & & & & \binom{r-1}{r-1} S_r(r-1) & + \quad (-1)^{2r-1} \binom{r}{r-1} S_r(r) \\
\varphi_r(r) = & & & & & & & & & & & & \binom{r}{r} S_r(r)
\end{array}$$

Let $\varphi_r = \left[\varphi_r(0), \varphi_r(1), \varphi_r(2), \ldots, \varphi_r(r)\right]^T$ and $S_r = [S_r(0), S_r(1), S_r(2), \ldots, S_r(r)]^T$ be column vectors of dimension r. Also, we define a matrix M_r. For this recall that $\binom{l}{k} = 0$ if $l, k \in \mathbb{Z}_+$ and $l < k$. Then M_r is given by

$$M_r = \begin{bmatrix} \binom{0}{0} & -\binom{1}{0} & \binom{2}{0} & \ldots & (-1)^{r+0}\binom{r}{0} \\ 0 & \binom{1}{1} & -\binom{2}{1} & \ldots & (-1)^{r+1}\binom{r}{1} \\ 0 & 0 & \binom{2}{2} & \ldots & (-1)^{r+2}\binom{r}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & (-1)^{2r}\binom{r}{r} \end{bmatrix}.$$

The entries below the diagonal are zero by definition and entries along the diagonal are equal to 1 by definition.

The system of equations that we obtained can be written more compactly in matrix notation,$\varphi_r = M_r S_r$. We claim that inverse of M_r exists, and that

$$M_r^{-1} = \begin{bmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \ldots & \binom{r}{0} \\ 0 & \binom{1}{1} & \binom{2}{1} & \ldots & \binom{r}{1} \\ 0 & 0 & \binom{2}{2} & \ldots & \binom{r}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \binom{r}{r} \end{bmatrix}.$$

We show this by multiplying M_r^{-1} and M_r which will result in the identity matrix $I_{(r+1)\times(r+1)}$. This then tells us that $M_r^{-1}\varphi_r = S_r$.

It suffices to show that the inner product of the j^{th} row of M_r with the k^{th} column of M_r^{-1} is equal to 0 if $j \neq k$ and 1 if $j = k$. The entry in the j row and k column is

$$[M_r M_r^{-1}]_{jk} = \sum_{l=0}^{r} (-1)^{j+l} \binom{l}{j}\binom{k}{l} = (-1)^j \binom{0}{j}\binom{k}{0} + (-1)^{j+1}\binom{1}{j}\binom{k}{1} + \cdots + (-1)^{j+l}\binom{l}{j}\binom{k}{l} + \cdots + (-1)^{j+r}\binom{r}{j}\binom{k}{r}.$$

Note, if $j = 0$, then $\binom{l}{j} = 1$. Now, if $k = 0$, $\binom{k}{l} = 1$ when $l = 0$ and $\binom{k}{l} = 0$ when $l > 0$. This means that when $j = 0$, $[M_r M_r^{-1}]_{jk} = 1$ if $k = 0$ and $[M_r M_r^{-1}]_{jk} = 0$ when $k > 0$. We can see that $\binom{n}{j} = 0$ for $0 \leq n < j$, thus $(-1)^j\binom{0}{j}\binom{k}{0} + (-1)^{j+1}\binom{1}{j}\binom{k}{1} + \cdots + (-1)^{j+l}\binom{l}{j}\binom{k}{l} + \cdots + (-1)^{j+r}\binom{r}{j}\binom{k}{r} = \binom{j}{j}\binom{k}{j} - \binom{j+1}{j}\binom{k}{j+1} + \cdots + (-1)^{j+l}\binom{l}{j}\binom{k}{l} + \cdots + (-1)^{j+r}\binom{r}{j}\binom{k}{r}$, where $j \leq l \leq r$. Notice that when $k \geq l \geq j$

$$\binom{l}{j}\binom{k}{l} = \frac{l!}{j!(l-j)!} \cdot \frac{k!}{l!(k-l)!} = \frac{1}{j!(l-j)!} \cdot \frac{k!}{(k-l)!} \cdot \frac{(k-j)!}{(k-j)!} = \frac{k!}{j!(k-j)!} \cdot \frac{(k-j)!}{(l-j)!(k-l)!} = \binom{k}{j}\binom{k-j}{l-j}.$$

Therefore,

$$\begin{aligned}\binom{j}{j}\binom{k}{j} - \binom{j+1}{j}\binom{k}{j+1} + \cdots + (-1)^{j+r}\binom{r}{j}\binom{k}{r} &= \binom{k}{j}\binom{k-j}{0} - \binom{k}{j}\binom{k-j}{1} + \cdots + (-1)^{j+l}\binom{k}{j}\binom{k-j}{l-j} + \cdots + (-1)^{j+r}\binom{k}{j}\binom{k-j}{r-j} \\ &= \binom{k}{j}\left[\binom{k-j}{0} - \binom{k-j}{1} + \cdots + (-1)^{j+l}\binom{k-j}{l-j} + \cdots + (-1)^{j+r}\binom{k-j}{r-j}\right].\end{aligned}$$

So,

$$[M_r M_r^{-1}]_{jk} = \binom{k}{j}\sum_{n=0}^{k-j}(-1)^n\binom{k-j}{n}.$$

The following three cases are needed to find the value of $[M_r M_r^{-1}]_{jk}$. If $k < j$, then $\binom{k}{l} = 0$ when $j \leq l \leq r$, so $\sum_{l=j}^{r}(-1)^{j+l}\binom{l}{j}\binom{k}{l} = 0$, thus $[M_r M_r^{-1}]_{jk} = 0$. If $k = j$, then $\binom{k-j}{n} = 0$ for all n except when $n = 0$. So, $[M_r M_r^{-1}]_{kk} = \binom{k}{k}\sum_{n=0}^{0}(-1)^n\binom{0}{n} = 1$. In the last case when $k > j$, using the binomial theorem we can see that $[M_r M_r^{-1}]_{jk} = \binom{k}{j}\sum_{n=0}^{k-j}(-1)^n\binom{k-j}{n} = \binom{k}{j}(1-1)^{k-j} = \binom{k}{j}0^{k-j} = 0$. Hence, every entry in the matrix product of $M_r M_r^{-1}$ is equal to 0 if $k \neq j$ and 1 if $k = j$. This means that M_r^{-1} is the inverse of M_r. Thus, $M_r^{-1}\varphi_r = S_r$. So,

$$\begin{bmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \dots & \binom{r}{0} \\ 0 & \binom{1}{1} & \binom{2}{1} & \dots & \binom{r}{1} \\ 0 & 0 & \binom{2}{2} & \dots & \binom{r}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \binom{r}{r} \end{bmatrix} \begin{bmatrix} \varphi_r(0) \\ \varphi_r(1) \\ \varphi_r(2) \\ \vdots \\ \varphi_r(r) \end{bmatrix} = \begin{bmatrix} \binom{0}{0}\varphi_r(0) + \binom{1}{0}\varphi_r(1) + \binom{2}{0}\varphi_r(2) + \cdots + \binom{r}{0}\varphi_r(r) \\ \binom{1}{1}\varphi_r(1) + \binom{2}{1}\varphi_r(2) + \cdots + \binom{r}{1}\varphi_r(r) \\ \binom{2}{2}\varphi_r(2) + \cdots + \binom{r}{2}\varphi_r(r) \\ \ddots \quad \vdots \\ \binom{r}{r}\varphi_r(r) \end{bmatrix} = \begin{bmatrix} S_r(0) \\ S_r(1) \\ S_r(2) \\ \vdots \\ S_r(r) \end{bmatrix}$$

This being the case, from (2.18) it follows that $\sum_{i=0}^{r}(-1)^i\sum_{j=i}^{r}\binom{j}{i}\varphi_r(j)\hat{\theta}_{r,\vec{p}}^{i} = 1 - \left(\sum_{i_1=1}^{r} p_{i_1}\right)\hat{\theta}_{r,\vec{p}} + \left(\sum_{1 \leq i_1 < i_2 \leq r} p_{i_1}p_{i_2}\right)\hat{\theta}_{r,\vec{p}}^{2} -$

$... + (-1)^r \left(\prod_{i_r=1}^r p_{i_r}\right) \hat{\theta}_{r,\vec{p}}^r.$ □

Corollary 3. *If* $\hat{\theta}_{r,\vec{p}} > 0$*, then*

$$0 = \sum_{i_1=1}^{r} p_{i_1} - 1 - \left(\sum_{1 \le i_1 < i_2 \le r} p_{i_1} p_{i_2} \right) \hat{\theta}_{r,\vec{p}} + ... - (-1)^r \left(\prod_{i_r=1}^{r} p_{i_r} \right) \hat{\theta}_{r,\vec{p}}^{r-1}. \tag{2.22}$$

Proof. Start with (2.16), subtract 1 from both sides, and add $\hat{\theta}_{r,\vec{p}}$ to both sides to get

$$0 = \hat{\theta}_{r,\vec{p}} - \left(\sum_{i_1=1}^{r} p_{i_1} \right) \hat{\theta}_{r,\vec{p}} - ... + (-1)^r \left(\prod_{i_r=1}^{r} p_{i_r} \right) \hat{\theta}_{r,\vec{p}}^{r}.$$

Finally divide by $-\hat{\theta}_{r,\vec{p}}$. □

Solving for $\hat{\theta}_{r,\vec{p}}$ in closed form as a function of $p_1, ..., p_r$ becomes increasingly complex as r increases. If a computer program is used, the closed form expression for $\hat{\theta}_{4,\vec{p}}$ is rather long to write out. Furthermore, for $r \ge 5$, there is no general form in these cases. Instead we develop properties of $\hat{\theta}_{r,\vec{p}}$ for all $r \ge 2$ using the the Implicit Function Theorem.

Corollary 4. *If* $\sum p_i > 1$ *and* $p_i < 1$ *for all* i*, then* $\hat{\theta}_{r,\vec{p}}$ *is differentiable as a function of the probability parameters* p_i*, thus continuous on some neighborhood of* $\vec{p} \in [0,1]^r$*.*

Proof. Suppose $\sum p_i > 1$ and $p_i < 1$ for all i. Then, by Theorem 2, $\hat{\theta}_{r,\vec{p}} > 0$. Hence, by Corollary 3, $\hat{\theta}_{r,\vec{p}}$ satisfies (2.22). Referring to the Implicit Function Theorem as stated in Appendix B, we set $G : \mathbb{R}^{r+1} \to \mathbb{R}$ to be

$$G(\mathbf{x}, y) = -\left(\sum_{i_1=1}^{r} x_{i_1} - 1 - \left(\sum_{1 \le i_1 < i_2 \le n} x_{i_1} x_{i_2} \right) y + ... - (-1)^r \left(\prod_{i_r=1}^{r} p_{i_r} \right) y^{r-1} \right),$$

for $\mathbf{x} \in \mathbb{R}^r$ and $y \in \mathbb{R}$. G is continuous and differentiable on $\mathbb{R}^{r+1}$ and $G\left(\vec{p}, \hat{\theta}_{r,\vec{p}}\right) = 0$ by Corollary 3 for all $\vec{p} \in [0,1]^r$ such that $\hat{\theta}_{r,\vec{p}} > 0$. Also,

$$\frac{\partial}{\partial y} G(\mathbf{x}, y) = \sum_{1 \le i_1 < i_2 \le r} x_{i_1} x_{i_2} + ... + (-1)^r (r-1) \left(\prod_{i_r=1}^{r} x_{i_r} \right) y^{r-2},$$

and it is necessary to show

$$\left. \frac{\partial}{\partial y} G(\mathbf{x}, y) \right|_{(\vec{p}, \hat{\theta}_{r,\vec{p}})} = \sum_{1 \le i_1 < i_2 \le r} p_{i_1} p_{i_2} + ... + (-1)^r (r-1) \left(\prod_{i_r=1}^{r} p_{i_r} \right) \hat{\theta}_{r,\vec{p}}^{r-2} \ne 0.$$

We will show that $\left. \frac{\partial}{\partial y} G(\mathbf{x}, y) \right|_{(\vec{p}, \hat{\theta}_{r,\vec{p}})} > 0$.

Before presenting a general argument that $\frac{\partial}{\partial y}G(\mathbf{x},y)\Big|_{(\vec{p},\hat{\theta}_{r,\vec{p}})} > 0$, for all $r \geq 2$. We first work out the $r = 4$ and $r = 5$ cases as indications of how the various terms in the expression can be carefully combined to show that the sum is always positive.

Let $r = 4$. For ease of notation in the calculations that follows, let $\hat{\theta} = \hat{\theta}_{4,\vec{p}}$

$$
\begin{aligned}
\frac{\partial}{\partial y}G(\mathbf{x},y)\Big|_{(\vec{p},\hat{\theta}_{4,\vec{p}})} =& p_1p_2 + p_1p_3 + p_1p_4 + p_2p_3 + p_2p_4 + p_3p_4 \\
& - 2p_1p_2p_3\hat{\theta} - 2p_1p_2p_4\hat{\theta} - 2p_1p_3p_4\hat{\theta} - 2p_2p_3p_4\hat{\theta} \\
& + 3p_1p_2p_3p_4\hat{\theta}^2 \\
=& p_1p_2 - p_1p_2p_3\hat{\theta} + p_1p_3 - p_1p_2p_3\hat{\theta} + p_1p_4 - p_1p_2p_4\hat{\theta} \\
& + p_2p_4 - p_1p_2p_4\hat{\theta} + p_2p_3 - p_2p_3p_4\hat{\theta} + p_3p_4 - p_1p_3p_4\hat{\theta} \\
& - p_1p_3p_4\hat{\theta} - p_2p_3p_4\hat{\theta} + 3p_1p_2p_3p_4\hat{\theta}^2 \\
=& p_1p_2(1-p_3\hat{\theta}) + p_1p_3(1-p_2\hat{\theta}) + p_1p_4(1-p_2\hat{\theta}) + p_2p_4(1-p_1\hat{\theta}) + p_2p_3(1-p_4\hat{\theta}) + p_3p_4(1-p_1\hat{\theta}) \\
& - p_1p_3p_4\hat{\theta}(1-p_2\hat{\theta}) - p_2p_3p_4\hat{\theta}(1-p_1\hat{\theta}) + p_1p_2p_3p_4\hat{\theta}^2 \\
=& p_1p_2(1-p_3\hat{\theta}) + p_1p_4(1-p_2\hat{\theta}) + p_2p_3(1-p_4\hat{\theta}) + p_3p_4(1-p_1\hat{\theta}) \\
& + p_1p_3(1-p_2\hat{\theta})(1-p_4\hat{\theta}) + p_2p_4(1-p_1\hat{\theta})(1-p_3\hat{\theta}) + p_1p_2p_3p_4\hat{\theta}^2.
\end{aligned}
$$

Note that each term is positive in the previous equality and thus $\frac{\partial}{\partial y}G(\mathbf{x},y)\Big|_{(\vec{p},\hat{\theta}_{r,\vec{p}})} > 0$. Therefore, $\hat{\theta}_{4,\vec{p}}$ exists and is continuous with variables p_1, p_2, p_3, p_4 on some neighborhood of $\vec{p} \in [0,1]^4$. In principle, we could have written an exact formula for $\hat{\theta}_{4,\vec{p}}$ since this satisfies a quartic equation. On the other hand, since there is no formula for the solution to a general quintic equation, we need to rely upon the Implicit Function Theorem to see that $\hat{\theta}_{5,\vec{p}}$ exists and is continuous on some open neighborhood. We show this by letting $r = 5$ and as before we let $\hat{\theta} = \hat{\theta}_{5,\vec{p}}$ in the calculation that follows.

$$
\begin{aligned}
\frac{\partial}{\partial y}G(\mathbf{x},y)\Big|_{(\vec{p},\hat{\theta}_{5,\vec{p}})} =& p_1p_2 + p_1p_3 + p_1p_4 + p_1p_5 + p_2p_3 + p_2p_4 + p_2p_5 + p_3p_4 + p_3p_5 + p_4p_5 \\
& - 2p_1p_2p_3\hat{\theta} - 2p_1p_2p_4\hat{\theta} - 2p_1p_2p_5\hat{\theta} - 2p_1p_3p_4\hat{\theta} - 2p_1p_3p_5\hat{\theta} - 2p_1p_4p_5\hat{\theta} \\
& - 2p_2p_3p_4\hat{\theta} - 2p_2p_3p_5\hat{\theta} - 2p_2p_4p_5\hat{\theta} - 2p_3p_4p_5\hat{\theta} + 3p_1p_2p_3p_4\hat{\theta}^2 + 3p_1p_2p_3p_5\hat{\theta}^2 \\
& + 3p_1p_2p_4p_5\hat{\theta}^2 + 3p_1p_3p_4p_5\hat{\theta}^2 + 3p_2p_3p_4p_5\hat{\theta}^2 - 4p_1p_2p_3p_4p_5\hat{\theta}^3
\end{aligned}
$$

$$
\begin{aligned}
=&p_1p_2-p_1p_2p_3\hat{\theta}+p_1p_3-p_1p_3p_5\hat{\theta}+p_1p_4-p_1p_2p_4\hat{\theta}+p_1p_5-p_1p_4p_5\hat{\theta}\\
&+p_2p_3-p_2p_3p_4\hat{\theta}+p_2p_4-p_1p_2p_4\hat{\theta}+p_2p_5-p_2p_3p_5\hat{\theta}\\
&+p_3p_4-p_3p_4p_5\hat{\theta}+p_3p_5-p_2p_3p_5\hat{\theta}+p_4p_5-p_1p_4p_5\hat{\theta}\\
&-p_1p_2p_5\hat{\theta}+p_1p_2p_3p_5\hat{\theta}^2-p_1p_3p_4\hat{\theta}+p_1p_3p_4p_5\hat{\theta}^2-p_1p_3p_4\hat{\theta}+p_1p_2p_3p_4\hat{\theta}^2\\
&-p_1p_2p_5\hat{\theta}+p_1p_2p_4p_5\hat{\theta}^2-p_1p_2p_3\hat{\theta}+p_1p_2p_3p_4\hat{\theta}^2-p_2p_4p_5\hat{\theta}+p_1p_2p_4p_5\hat{\theta}^2\\
&-p_2p_4p_5\hat{\theta}+p_2p_3p_4p_5\hat{\theta}^2-p_2p_3p_4\hat{\theta}+p_2p_3p_4p_5\hat{\theta}^2-p_1p_3p_5\hat{\theta}+p_1p_2p_3p_5\hat{\theta}^2\\
&-p_3p_4p_5\hat{\theta}+p_1p_3p_4p_5\hat{\theta}^2\\
&+p_1p_2p_3p_4\hat{\theta}^2-p_1p_2p_3p_4p_5\hat{\theta}^3+p_1p_2p_3p_5\hat{\theta}^2-p_1p_2p_3p_4p_5\hat{\theta}^3\\
&+p_1p_2p_4p_5\hat{\theta}^2-p_1p_2p_3p_4p_5\hat{\theta}^3+p_1p_3p_4p_5\hat{\theta}^2-p_1p_2p_3p_4p_5\hat{\theta}^3+p_2p_3p_4p_5\hat{\theta}^2\\
=&p_1p_2(1-p_3\hat{\theta})+p_1p_3(1-p_5\hat{\theta})+p_1p_4(1-p_2\hat{\theta})+p_1p_5(1-p_4\hat{\theta})\\
&+p_2p_3(1-p_4\hat{\theta})+p_2p_4(1-p_1\hat{\theta})+p_2p_5(1-p_3\hat{\theta})\\
&+p_3p_4(1-p_5\hat{\theta})+p_3p_5(1-p_2\hat{\theta})+p_4p_5(1-p_1\hat{\theta})\\
&-p_1p_2p_5\hat{\theta}(1-p_3\hat{\theta})-p_1p_3p_4\hat{\theta}(1-p_5\hat{\theta})-p_1p_3p_4\hat{\theta}(1-p_2\hat{\theta})\\
&-p_1p_2p_5\hat{\theta}(1-p_4\hat{\theta})-p_1p_2p_3\hat{\theta}(1-p_4\hat{\theta})-p_2p_4p_5\hat{\theta}(1-p_1\hat{\theta})\\
&-p_2p_4p_5\hat{\theta}(1-p_3\hat{\theta})-p_2p_3p_4\hat{\theta}(1-p_5\hat{\theta})-p_1p_3p_5\hat{\theta}(1-p_2\hat{\theta})-p_3p_4p_5\hat{\theta}(1-p_1\hat{\theta})\\
&+p_1p_2p_3p_4\hat{\theta}^2(1-p_5\hat{\theta})+p_1p_2p_3p_5\hat{\theta}^2(1-p_4\hat{\theta})+p_1p_2p_4p_5\hat{\theta}^2(1-p_3\hat{\theta})\\
&+p_1p_3p_4p_5\hat{\theta}^2(1-p_2\hat{\theta})+p_2p_3p_4p_5\hat{\theta}^2\\
=&p_1p_2(1-p_3\hat{\theta})(1-p_5\hat{\theta})+p_1p_3(1-p_5\hat{\theta})(1-p_4\hat{\theta})+p_1p_4(1-p_2\hat{\theta})(1-p_3\hat{\theta})\\
&+p_1p_5(1-p_4\hat{\theta})(1-p_2\hat{\theta})+p_2p_3(1-p_4\hat{\theta})(1-p_1\hat{\theta})+p_2p_4(1-p_1\hat{\theta})(1-p_5\hat{\theta})\\
&+p_2p_5(1-p_3\hat{\theta})(1-p_4\hat{\theta})+p_3p_4(1-p_5\hat{\theta})(1-p_2\hat{\theta})+p_3p_5(1-p_2\hat{\theta})(1-p_1\hat{\theta})\\
&+p_4p_5(1-p_1\hat{\theta})(1-p_3\hat{\theta})\\
&+p_1p_2p_3p_4\hat{\theta}^2(1-p_5\hat{\theta})+p_1p_2p_3p_5\hat{\theta}^2(1-p_4\hat{\theta})+p_1p_2p_4p_5\hat{\theta}^2(1-p_3\hat{\theta})\\
&+p_1p_3p_4p_5\hat{\theta}^2(1-p_2\hat{\theta})+p_2p_3p_4p_5\hat{\theta}^2.
\end{aligned}
$$

Note that every term in the last equality is positive and thus $\frac{\partial}{\partial y}G(\mathbf{x},y)\Big|_{(\vec{p},\hat{\theta}_{5,\vec{p}})}>0$.

It is easily seen that $\frac{\partial}{\partial y}G(\mathbf{x},y)\Big|_{(\vec{p},\hat{\theta}_{r,\vec{p}})}>0$ in the case where $r=2,3$. We have just shown that this is also true for $r=4$ and $r=5$, but our argument shows that this becomes increasingly more complicated to see as r increases if we continue to try to simply regroup terms.

It turns out that we can prove that $\frac{\partial}{\partial y}G(\mathbf{x},y)\Big|_{(\vec{p},\hat{\theta}_{r,\vec{p}})} > 0$ for general r by mathematical induction. We assume $\frac{\partial}{\partial y}G(\mathbf{x},y)\Big|_{(\vec{p},\hat{\theta}_{r,\vec{p}})} > 0$ for some $r \geq 5$ and we seek to show that $\frac{\partial}{\partial y}G(\mathbf{x},y)\Big|_{(\vec{p},\hat{\theta}_{r+1,\vec{p}})} > 0$. Again, we let $\hat{\theta} = \hat{\theta}_{r+1,\vec{p}}$ for simplicity. We begin by expanding $\frac{\partial}{\partial y}G(\mathbf{x},y)\Big|_{(\vec{p},\hat{\theta}_{r+1,\vec{p}})}$ and then separating the terms into those that don't contain p_{r+1} and those that do contain this term.

$$
\begin{aligned}
\frac{\partial}{\partial y}G(\mathbf{x},y)\Big|_{(\vec{p},\hat{\theta}_{r+1,\vec{p}})} =& p_1p_2 + p_1p_3 + \ldots + p_1p_{r+1} + p_2p_3 + \ldots + p_2p_{r+1} + \ldots + p_rp_{r+1} \\
& - 2p_1p_2p_3\hat{\theta} - 2p_1p_2p_4\hat{\theta} - \ldots - 2p_1p_rp_{r+1}\hat{\theta} - 2p_2p_3p_4\hat{\theta} - \ldots - 2p_{r-1}p_rp_{r+1}\hat{\theta} \\
& + 3p_1p_2p_3p_4\hat{\theta}^2 + \ldots + 3p_1p_2p_3p_{r+1}\hat{\theta}^2 + \ldots + 3p_{r-2}p_{r-1}p_rp_{r+1}\hat{\theta}^2 \\
& \qquad \vdots \\
& + (-1)^r(r-1)p_1p_2\ldots p_r\hat{\theta}^{r-2} + (-1)^r(r-1)p_1p_2\ldots p_{r-1}p_{r+1}\hat{\theta}^{r-2} \\
& + \ldots + (-1)^r(r-1)p_1p_3\ldots p_rp_{r+1}\hat{\theta}^{r-2} + (-1)^r(r-1)p_2p_3\ldots p_rp_{r+1}\hat{\theta}^{r-2} \\
& + (-1)^{r+1}(r)p_1p_2p_3\ldots p_{r-1}p_rp_{r+1}\hat{\theta}^{r-1} \\
=& \Big[p_1p_2 + p_1p_3 + \ldots + p_1p_r + p_2p_3 + \ldots + p_2p_r + \ldots + p_{r-1}p_r \\
& - 2p_1p_2p_3\hat{\theta} - 2p_1p_2p_4\hat{\theta} - \ldots - 2p_1p_{r-1}p_r\hat{\theta} - 2p_2p_3p_4\hat{\theta} - \ldots - 2p_{r-2}p_{r-1}p_r\hat{\theta} \\
& + 3p_1p_2p_3p_4\hat{\theta}^2 + \ldots + 3p_1p_2p_3p_r\hat{\theta}^2 + \ldots + 3p_{r-3}p_{r-2}p_{r-1}p_r\hat{\theta}^2 \\
& \qquad \vdots \\
& + (-1)^r(r-1)p_1p_2\ldots p_{r-1}p_r\hat{\theta}^{r-2}\Big] \\
& + p_1p_{r+1} + p_2p_{r+1} + \ldots + p_rp_{r+1} \\
& - 2p_1p_2p_{r+1}\hat{\theta} - 2p_1p_3p_{r+1}\hat{\theta} - \ldots - 2p_1p_rp_{r+1}\hat{\theta} - \ldots - 2p_{r-1}p_rp_{r+1}\hat{\theta} \\
& + 3p_1p_2p_3p_{r+1}\hat{\theta}^2 + \ldots + 3p_1p_2p_rp_{r+1}\hat{\theta}^2 + \ldots + 3p_{r-2}p_{r-1}p_rp_{r+1}\hat{\theta}^2 \\
& \qquad \vdots \\
& + (-1)^r(r-1)p_1p_2\ldots p_{r-1}p_{r+1}\hat{\theta}^{r-2} \\
& + \ldots + (-1)^r(r-1)p_1p_3\ldots p_rp_{r+1}\hat{\theta}^{r-2} + (-1)^r(r-1)p_2p_3\ldots p_rp_{r+1}\hat{\theta}^{r-2} \\
& + (-1)^{r+1}(r)p_1p_2p_3\ldots p_{r-1}p_rp_{r+1}\hat{\theta}^{r-1}.
\end{aligned}
$$

Note that the collective sum and difference of all the terms between the brackets that do not have a p_{r+1} factor is positive by our assumption. We will refer to this collection of sum and difference of these terms as Λ. Next we will factor out a p_{r+1} from all the terms that are not part of Λ and then observe that we can group those remaining terms to reveal another $-\hat{\theta}\Lambda$ inside that expression.

$$
\begin{aligned}
\frac{\partial}{\partial y}G(\mathbf{x},y)\Big|_{(\vec{p},\hat{\theta}_{r+1},\vec{p})} =& \Lambda + p_{r+1}\Big[p_1+p_2+\ldots+p_r \\
& -2p_1p_2\hat{\theta}-2p_1p_3\hat{\theta}-\ldots-2p_1p_r\hat{\theta}-\ldots-2p_{r-1}p_r\hat{\theta} \\
& +3p_1p_2p_3\hat{\theta}^2+\ldots+3p_1p_2p_r\hat{\theta}^2+\ldots+3p_{r-2}p_{r-1}p_r\hat{\theta}^2 \\
& \vdots \\
& +(-1)^r(r-1)p_1p_2\ldots p_{r-1}\hat{\theta}^{r-2} \\
& +\ldots+(-1)^r(r-1)p_1p_3\ldots p_r\hat{\theta}^{r-2}+(-1)^r(r-1)p_2p_3\ldots p_r\hat{\theta}^{r-2} \\
& +(-1)^{r+1}(r)p_1p_2p_3\ldots p_{r-1}p_r\hat{\theta}^{r-1}\Big] \\
=& \Lambda + p_{r+1}\Big[p_1+p_2+\ldots+p_r-p_1p_2\hat{\theta}-p_1p_3\hat{\theta}-\ldots-p_1p_r\hat{\theta}-\ldots-p_{r-1}p_r\hat{\theta} \\
& +p_1p_2p_3\hat{\theta}^2+\ldots+p_1p_2p_r\hat{\theta}^2+\ldots+p_{r-2}p_{r-1}p_r\hat{\theta}^2 \\
& +(-1)^rp_1p_2\ldots p_{r-1}\hat{\theta}^{r-2}+\ldots+(-1)^rp_1p_3\ldots p_r\hat{\theta}^{r-2}+(-1)^rp_2p_3\ldots p_r\hat{\theta}^{r-2} \\
& +(-1)^{r+1}p_1p_2p_3\ldots p_{r-1}p_r\hat{\theta}^{r-1} \\
& -p_1p_2\hat{\theta}-p_1p_3\hat{\theta}-\ldots-p_1p_r\hat{\theta}-\ldots-p_{r-1}p_r\hat{\theta} \\
& +2p_1p_2p_3\hat{\theta}^2+\ldots+2p_1p_2p_r\hat{\theta}^2+\ldots+2p_{r-2}p_{r-1}p_r\hat{\theta}^2 \\
& \vdots \\
& +(-1)^r(r-2)p_1p_2\ldots p_r\hat{\theta}^{r-2} \\
& +\ldots+(-1)^r(r-1)p_1p_3\ldots p_r\hat{\theta}^{r-2}+(-1)^r(r-1)p_2p_3\ldots p_r\hat{\theta}^{r-2} \\
& +(-1)^{r+1}(r-1)p_1p_2p_3\ldots p_{r-1}p_r\hat{\theta}^{r-1}\Big] \\
=& \Lambda + p_{r+1}\Big[p_1+p_2+\ldots+p_r-p_1p_2\hat{\theta}-p_1p_3\hat{\theta}-\ldots-p_1p_r\hat{\theta}-\ldots-p_{r-1}p_r\hat{\theta} \\
& +p_1p_2p_3\hat{\theta}^2+\ldots+p_1p_2p_r\hat{\theta}^2+\ldots+p_{r-2}p_{r-1}p_r\hat{\theta}^2 \\
& +(-1)^rp_1p_2\ldots p_{r-1}\hat{\theta}^{r-2}+\ldots+(-1)^rp_1p_3\ldots p_r\hat{\theta}^{r-2}+(-1)^rp_2p_3\ldots p_r\hat{\theta}^{r-2} \\
& +(-1)^{r+1}p_1p_2p_3\ldots p_{r-1}p_r\hat{\theta}^{r-1} \\
& -\hat{\theta}\Big[p_1p_2+p_1p_3+\ldots+p_1p_r+\ldots+p_{r-1}p_r \\
& -2p_1p_2p_3\hat{\theta}-\ldots-2p_1p_2p_r\hat{\theta}-\ldots-2p_{r-2}p_{r-1}p_r\hat{\theta} \\
& \vdots \\
& +(-1)^{r-1}(r-2)p_1p_2\ldots p_{r-1}\hat{\theta}^{r-3} \\
& +\ldots+(-1)^{r-1}(r-2)p_1p_3\ldots p_{r-1}\hat{\theta}^{r-3}+(-1)^{r-1}(r-2)p_2p_3\ldots p_r\hat{\theta}^{r-3} \\
& +(-1)^r(r-1)p_1p_2p_3\ldots p_{r-1}p_r\hat{\theta}^{r-2}\Big]\Big].
\end{aligned}
$$

Once again we can see that the terms between the inner brackets can be replaced by with Λ. Therefore,

$$\begin{aligned}\frac{\partial}{\partial y}G(\mathbf{x},y)\Big|_{(\vec{p},\hat{\theta}_{r+1,\vec{p}})} =&\Lambda + p_{r+1}\Big[p_1 + p_2 + ... + p_r - p_1p_2\hat{\theta} - p_1p_3\hat{\theta} - ... - p_1p_r\hat{\theta} - ... - p_{r-1}p_r\hat{\theta}\\ &+ p_1p_2p_3\hat{\theta}^2 + ... + p_1p_2p_r\hat{\theta}^2 + ... + p_{r-2}p_{r-1}p_r\hat{\theta}^2\\ &+ (-1)^r p_1p_2...p_{r-1}\hat{\theta}^{r-2} + ... + (-1)^r p_1p_3...p_r\hat{\theta}^{r-2} + (-1)^r p_2p_3...p_r\hat{\theta}^{r-2}\\ &+ (-1)^{r+1}p_1p_2p_3...p_{r-1}p_r\hat{\theta}^{r-1}\Big] - \hat{\theta}p_{r+1}\Lambda\end{aligned}$$

The remaining terms in between the brackets can be rewritten as $\dfrac{1-(1-p_1)(1-p_2)...(1-p_r)}{\hat{\theta}}$ which is positive. Thus

$$\begin{aligned}\frac{\partial}{\partial y}G(\mathbf{x},y)\Big|_{(\vec{p},\hat{\theta}_{r+1,\vec{p}})} =&\Lambda - \hat{\theta}p_{r+1}\Lambda + p_{r+1}\left[\frac{1-(1-p_1)(1-p_2)...(1-p_r)}{\hat{\theta}}\right]\\ =&\Lambda(1-\hat{\theta}p_{r+1}) + p_{r+1}\left[\frac{1-(1-p_1)(1-p_2)...(1-p_r)}{\hat{\theta}}\right] > 0.\end{aligned}$$

Therefore,

$$\frac{\partial}{\partial y}G(\mathbf{x},y)\Big|_{(\vec{p},\hat{\theta}_{r+1,\vec{p}})} > 0.$$

□

Having looked at the general case, we can now move on the full regular tree.

2.6 Percolation on Regular Trees

In this section, we explorer the $(r+1)$-regular trees. We will use the knowledge gained in the previous sections to find the percolation probability of a few chosen vertices on regular tree formed as follows. Taking two identical r-ary trees, one with root $\varnothing$ and the other tree with root $\overline{\varnothing}$ and adding an edge between the two roots, we form a an $(r+1)$-regular tree. If we let the edges of each r-ary subgraph have the probability of being open as in Section 2.3.2, while allowing the edge between the roots have the probability p of being open we form an $(r+1)$-regular tree. The 3-regular tree is shown in Figure 2.7 .

Let $\theta^v_{r+1}(\vec{p})$ = probability that percolation occurs where an infinite path begins at vertex v in the $(r+1)$-

regular tree for $\vec{p} = (p, p_1, p_2, ..., p_r)$ where probability that the edge between $\emptyset$ and $\overline{\emptyset}$ is open with probability p and where the other parameters are the probabilities of the edges being open for the r-ary subgraphs. Note that the absence of the $\hat{}$ above the θ corresponds to the regular tree to distinguish it from the r-ary tree. In what follows, the value of p will be equal to either p_1 or p_2. We may also write $\theta^v_{r+1}(\vec{p})$ as $\theta^v_{(r+1),\vec{p}}$.

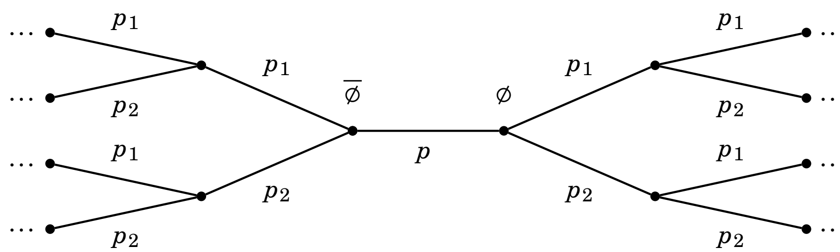

Figure 2.7: The 3-Regular Tree with probability parameters $\vec{p} = (p, p_1, p_2)$.

Lets first consider $r = 2$ in the homogeneous case where, i.e., $p_1 = p_2 = p$. The probability of not having percolation at the chosen vertex $\emptyset$ is $1 - \theta^{\emptyset}_{3,\vec{p}}$. Note that if the edge between $\emptyset$ and $\overline{\emptyset}$ is removed, percolation does not occur on the binary subgraph with root $\emptyset$ with probability $1 - \hat{\theta}^{\emptyset}_{2,\vec{p}}$. If we look at the subgraph that includes the binary subgraph with root $\overline{\emptyset}$, the edge with probability p and the vertex $\emptyset$, not percolating on it is $1 - p\hat{\theta}^{\emptyset}_{2,\vec{p}}$. To make the notation simpler we let $\theta^v = \theta^v_{3,\vec{p}}$ and $\hat{\theta}^u = \hat{\theta}^u_{2,\vec{p}}$ or in the case $u = \emptyset$ or $\overline{\emptyset}$, we write $\hat{\theta}$. Therefore we have,

$$1 - \theta^{\emptyset} = (1 - \hat{\theta})(1 - p\hat{\theta}) = 1 - \hat{\theta} - p\hat{\theta} + p\hat{\theta}^2. \tag{2.23}$$

Solving for $\theta^{\emptyset}$ and substituting $\hat{\theta} = \frac{2p-1}{p^2}$ we get the percolation probability as a function of p when $p \geq \frac{1}{2}$,

$$\begin{aligned} \theta^{\emptyset} &= \hat{\theta} + p\hat{\theta} - p\hat{\theta}^2 \qquad (2.24)\\ &= \frac{2p-1}{p^2} + p\left(\frac{2p-1}{p^2}\right) - p\left(\frac{2p-1}{p^2}\right)^2 \\ &= \frac{2p^3 - 3p^2 + 3p - 1}{p^3}. \qquad (2.25) \end{aligned}$$

Therefore the homogeneous percolation probability function is given by,

$$\theta^{\emptyset}_3 = \begin{cases} 0, & 0 \leq p < \frac{1}{2}, \\ \frac{2p^3 - 3p^2 + 3p - 1}{p^3}, & \frac{1}{2} \leq p \leq 1. \end{cases}$$

Now we move to some inhomogeneous cases. On the infinite 3-regular inhomogeneous tree case, the edge between $\emptyset$ and $\overline{\emptyset}$ could have the probability p_1 or p_2 of being open. Choosing $p = p_1$ as our probability, (2.24) becomes,

$$1-\theta^{\emptyset} = (1-\hat{\theta})(1-p_1\hat{\theta}) = 1-\hat{\theta}-p_1\hat{\theta}+p_1\hat{\theta}^2.$$

Substituting $\hat{\theta} = \frac{p_1+p_2-1}{p_1p_2}$ and solving for $\theta^{\emptyset}$ as a function of p_1 and p_2,

$$\begin{aligned}
\theta^{\emptyset} =& \hat{\theta}+p_1\hat{\theta}-p_1\hat{\theta}^2 \qquad (2.26)\\
=& \frac{p_1+p_2-1}{p_1p_2}+p_1\left(\frac{p_1+p_2-1}{p_1p_2}\right)-p_1\left(\frac{p_1+p_2-1}{p_1p_2}\right)^2\\
=& \frac{p_1+p_2-1}{p_1p_2}+\frac{p_1+p_2-1}{p_2}-\frac{(p_1+p_2-1)^2}{p_1p_2^2}\\
=& \frac{p_2(p_1+p_2-1)}{p_1p_2^2}+\frac{p_1p_2(p_1+p_2-1)}{p_1p_2^2}\\
&-\frac{p_1^2+2p_1p_2+p_2^2-2(p_1+p_2)+1}{p_1p_2^2}\\
=& \frac{p_1p_2+p_2^2-p_2+p_1^2p_2+p_1p_2^2-p_1p_2}{p_1p_2^2}\\
&+\frac{-p_1^2-2p_1p_2-p_2^2+2(p_1+p_2)-1}{p_1p_2^2}\\
=& \frac{p_1^2p_2+p_1p_2^2-2p_1p_2-p_1^2+2p_1+p_2-1}{p_1p_2^2} \qquad (2.27)
\end{aligned}$$

Therefore, the percolation probability function for the 3-regular tree inhomogeneous case where the bond between $\emptyset$ and $\overline{\emptyset}$ is open with probability p_1 is given by

$$\theta_3^{\emptyset} = \begin{cases} 0, & 0 \le p_1+p_2 < 1, \\ \frac{p_1^2p_2+p_1p_2^2-2p_1p_2-p_1^2+2p_1+p_2-1}{p_1p_2^2}, & 1 \le p_1+p_2, p_1 \neq 0, p_2 \neq 0. \end{cases} \qquad (2.28)$$

Likewise, upon reversing the roles of p_1 and p_2 in (2.28), the percolation probability function in the 3-regular tree inhomogeneous case where the bond between $\emptyset$ and $\overline{\emptyset}$ is open with probability p_2 is,

$$\theta_3^{\emptyset} = \begin{cases} 0, & 0 \le p_1+p_2 < 1, \\ \frac{p_1^2p_2+p_1p_2^2-2p_1p_2-p_2^2+p_1+2p_2-1}{p_1^2p_2}, & 1 \le p_1+p_2, p_1 \neq 0, p_2 \neq 0. \end{cases}$$

In particular, if $p_1 = p_2$, then line (2.27) becomes line (2.25).

We may generalize line (2.26) to the full inhomogeneous (r+1)-regular tree, $r \ge 2$, but since $\hat{\theta}$ will most likely be complicated to write out in terms of $p, p_1, ..., p_r$, we simply write,

$$\theta_3^{\emptyset} = \begin{cases} 0, & 0 \le p_1 + ... + p_r < 1 \\ \hat{\theta} + p\hat{\theta} - p\hat{\theta}^2, & 1 \le p_1 + ... + p_r. \end{cases}$$

It is natural to ask what the probability of percolation starting from a different vertex other than $\overline{\emptyset}$ and $\emptyset$ is in the inhomogeneous case.

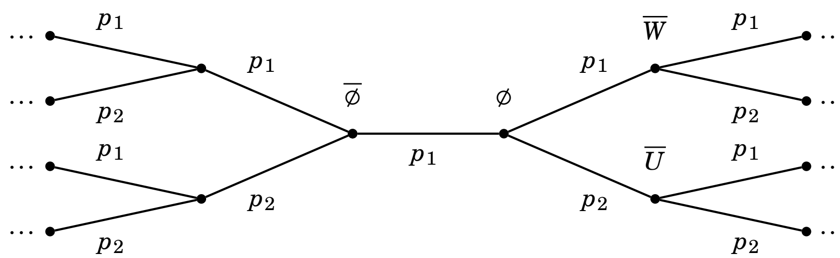

Figure 2.8: The 3-Regular Tree with edge probabilities parameters $\vec{p} = (p, p_1, p_2)$ and vertices at distance 1 from $\emptyset$ labeled.

Let's look at an example and calculate the probability that $\overline{U}$ percolates, where $\overline{U}$ is as indicated in Figure 2.8. Vertex $\overline{U}$ can be thought as the root of a binary tree just like $\emptyset$. Deleting the edge between $\emptyset$ and $\overline{U}$, we see that not percolating only on the binary subgraph where $\overline{U}$ is the root is $1 - \hat{\theta}$. Let F be the event that there exists an infinite open path where $\emptyset$ is part of this path, but the edge between $\emptyset$ and $\overline{U}$ is not used. We let $\mathbb{P}(F)$ be the probability of F. The probability of not percolating through the edge between $\emptyset$ and $\overline{U}$ is $1 - p_2\mathbb{P}(F)$. Not percolating through the edge between $\emptyset$ and $\overline{\emptyset}$ and the subgraph where $\overline{\emptyset}$ is the root of a binary tree when the edge between $\emptyset$ and $\overline{\emptyset}$ is deleted, is $1 - p_1\hat{\theta}$. Similarly, we can see that not percolating through the edge between $\emptyset$ and $\overline{W}$ and the subgraph where $\overline{W}$ is the root when the edge between $\overline{W}$ and $\emptyset$ is deleted, is $1 - p_1\hat{\theta}$. Therefore we can see that $\mathbb{P}(F) = 1 - (1 - p_1\hat{\theta})^2$. Thus, the probability of not percolating through the edge with endpoints $\emptyset$ and $\overline{U}$ is $1 - p_2\left(1 - (1 - p_1\hat{\theta})^2\right)$. Since percolating through the subgraph of where $\overline{U}$ is the root and percolating through the edge with endpoints $\emptyset$ and $\overline{U}$ are independent events, if follows that,

$$\begin{aligned}
1-\theta^{\overline{U}} &= (1-\hat{\theta})\left[1-p_2(1-(1-p_1\hat{\theta})^2)\right] \\
&= (1-\hat{\theta})(1-p_2(p_1\hat{\theta})(2-p_1\hat{\theta})) \\
&= (1-\hat{\theta})(1-2p_1p_2\hat{\theta}+p_1^2p_2\hat{\theta}^2) \\
&= 1-2p_1p_2\hat{\theta}+p_1^2p_2\hat{\theta}^2-\hat{\theta}+2p_1p_2\hat{\theta}^2-p_1^2p_2\hat{\theta}^3 \\
&= 1-(2p_1p_2+1)\hat{\theta}+(p_1^2p_2+2p_1p_2)\hat{\theta}^2-p_1^2p_2\hat{\theta}^3.
\end{aligned}$$

Thus upon substituting the formula for $\hat{\theta}$ that applies when $p_1+p_2 \geq 1$ and $p_1, p_2 > 0$, we obtain $\theta^{\overline{U}} = p_1^2p_2\hat{\theta}^3 - (p_1^2p_2+2p_1p_2)\hat{\theta}^2+(2p_1p_2+1)\hat{\theta}$, which is a more complicated expression.

$$\begin{aligned}
\theta^{\overline{U}} &= p_1^2p_2\hat{\theta}^3-(p_1^2p_2+2p_1p_2)\hat{\theta}^2+(2p_1p_2+1)\hat{\theta} \\
&= p_1^2p_2\left(\frac{(p_1+p_2-1)^3}{p_1^3p_2^3}\right)-(p_1^2p_2+2p_1p_2)\left(\frac{(p_1+p_2-1)^2}{p_1^2p_2^2}\right)+(2p_1p_2+1)\left(\frac{p_1+p_2-1}{p_1p_2}\right) \\
&= \frac{(p_1+p_2-1)^3}{p_1p_2^2}-(p_1+2)\left(\frac{(p_1+p_2-1)^2}{p_1p_2}\right)+(2p_1p_2+1)\left(\frac{p_1+p_2-1}{p_1p_2}\right) \\
&= \frac{(p_1+p_2-1)^3}{p_1p_2^2}-\frac{(p_2(p_1+2))(p_1+p_2-1)^2}{p_1p_2^2}+\frac{p_2(2p_1p_2+1)(p_1+p_2-1)}{p_1p_2^2} \\
&= \frac{(p_1+p_2-1)\left[(p_1+p_2-1)^2+(p_2(p_1+2))(p_1+p_2-1)+p_2(2p_1p_2+1)\right]}{p_1p_2^2},
\end{aligned}$$

for $p_1, p_2 > 0$ and $p_1+p_2 \geq 1$. This foreshadows the possibility that no explicit easily obtained expression may exist for the percolation probability for an arbitrary chosen vertex of a $(r+1)$-regular tree with inhomogeneous bond percolation as considered in this section. More research is needed to achieve further general results.

Chapter 3

Percolation with Periodic Inhomogeneities (of Period N)

3.1 Introduction to the Square Lattice

For the rest of the thesis, we take $\mathbb{Z}^2$ to be our set of vertices, where $\mathbb{Z}$ is the set of integers. The vertices in $\mathbb{Z}^2$ are ordered pairs of the form $v = (v_1, v_2)$, where $v_1, v_2 \in \mathbb{Z}$. The distance on $\mathbb{Z}^2$ will be defined as $\delta(u,v) = \sum_{i=1}^{2} |u_i - v_i|$. This is the l_1 metric where u_i is the i^{th} coordinate of u and v_i is the i^{th} coordinate of v. The *square lattice*, $\mathbb{L}^2$, is the graph whose vertex set $\mathbf{V}(\mathbb{L}^2)$ is the set of vertices in $\mathbb{Z}^2$ and whose edge set $\mathbf{E}(\mathbb{L}^2)$ is the set of edges of nearest neighbor edges defined as follows. A nearest neighbor edge in $\mathbb{L}^2$ is a pair of endpoints u and v where $u, v \in \mathbb{Z}^2$ and $\delta(u,v) = 1$. If the endpoints of an edge are u, v, then the edge can be represented by either $\langle u, v\rangle$ or $\langle v, u\rangle$. Note that our edges are undirected, so $\langle u, v\rangle = \langle v, u\rangle$. Figure 3.2 is a representation of a subset of $\mathbb{L}^2$ with the origin $(0,0)$ labeled as **O**.

We now define a "face" of a graph. For the graph illustrated in Figure 3.2, the faces will be the open unit squares of the form $(n, n+1) \times (m, m+1)$, where n and m are integers. For the definition of a face, we follow [1]. A *polygonal u,v curve* is a continuous map from $[0,1]$ to $\mathbb{R}^2$ satisfying the following conditions: its values at 0 and 1 are u and v respectively, and its image consists of a finite number of line segments. A *drawing* of a graph $\mathbf{G}$ is a function f defined on $\mathbf{V}(\mathbf{G}) \cup \mathbf{E}(\mathbf{G})$ that assigns each vertex u to a point $f(u)$ in $\mathbb{R}^2$ and assigns each edge with endpoints u, v to a polygonal $f(u), f(v)$-curve. A *crossing* is a common point in the image of a drawing of two distinct edges that is not the image of endpoints of these edges. If a graph **G** has a drawing without crossings of edges, then it is called a *planar* graph and we call that drawing a *planar embedding* of **G**. We use the usual topology on $\mathbb{R}^2$ and declare that a set $U \subseteq \mathbb{R}^2$ is *open* if for all $x \in U$, there exists an $\varepsilon > 0$ such that $\{y \in \mathbb{R}^2 \mid ||x-y|| < \varepsilon\} \subseteq U$, where for $x, y \in \mathbb{R}^2$, $||x-y|| = (|x_2-x_1|^2 + |y_2-y_1|^2)^{1/2}$. Note that this use of open is different from our use in connection with the bonds in our percolation model. An open set $U \subseteq \mathbb{R}^2$ that contains a polygonal u,v*-curve* for all $u,v \in U$ is called a *region*. A *face* of a planar graph is the maximal region of the plane that contains no point used in the embedding [1].

Having defined a face of a graph, we can now define the *dual* of a planar graph. Let **G** be a planar graph, to which we associate another graph $\mathbf{G}_\mathrm{D}$ as follows. We associate a vertex of $\mathbf{G}_\mathrm{D}$ to each face of **G**. The edges of $\mathbf{G}_\mathrm{D}$ have endpoints which are pairs of vertices of $\mathbf{G}_\mathrm{D}$ associated with two faces of **G** that share an edge of **G** as a boundary. There is a natural correspondence from the edges of $\mathbf{G}_\mathrm{D}$ to the edges of **G**, as illustrated in Figure 3.1. Vertices and edges of **G** are filled/solid and the vertices and edges of $\mathbf{G}_\mathrm{D}$ are open/dashed. Note that we have not indicated the dual vertex in the infinite face which is outside the hexagon.

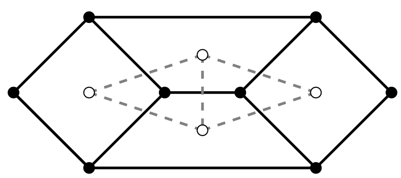

Figure 3.1: Graph and its Dual

We remark that if $\mathbf{G}_\mathrm{D}$ is the dual of **G**, then the dual of $\mathbf{G}_\mathrm{D}$, i.e. $(\mathbf{G}_\mathrm{D})_\mathrm{D}$, is **G**. So duality is a symmetric operation.

3.2 Bond Percolation on $\mathbb{L}^2$

As in the Chapter 1, we associate a probability $p_{\langle u,v\rangle}$, $0 \le p_{\langle u,v\rangle} \le 1$ with each edge $\langle u,v\rangle$ of $\mathbb{L}^2$. The terms *path*, *endpoint*, and length of a path are defined as in Section 1.1. The terms *open*, *closed*, *open path*, the sample space, *bond percolation* or just *percolation*, and *percolation probability* are also defined as in Section 1.2. All edges $\langle u,v\rangle$ in $\mathbb{L}^2$ will be either open or closed, but not both. An edge or *bond*, $\langle u,v\rangle$, is said to be *on* if the associated probability $p_{\langle u,v\rangle} = 1$ and is said to be *off* if the associated probability $p_{\langle u,v\rangle} = 0$. The *percolation probability* of a chosen vertex v is denoted $\Theta^v_{\mathbb{L}^2}(\mathbf{p})$, where $\mathbf{p} = (p_{\langle u,v\rangle}); \langle u,v\rangle \in \mathbf{E}(\mathbb{L}^2)$. Likewise, *homogeneous* bond percolation and *inhomogeneous* bond percolation are also defined similarly as in Chapter 1. We will let $\Theta_{\mathbb{L}^2}(\mathbf{p}) = \Theta^O_{\mathbb{L}^2}(\mathbf{p})$, where $\mathbf{O} = (0,0)$ is the origin on $\mathbb{L}^2$.

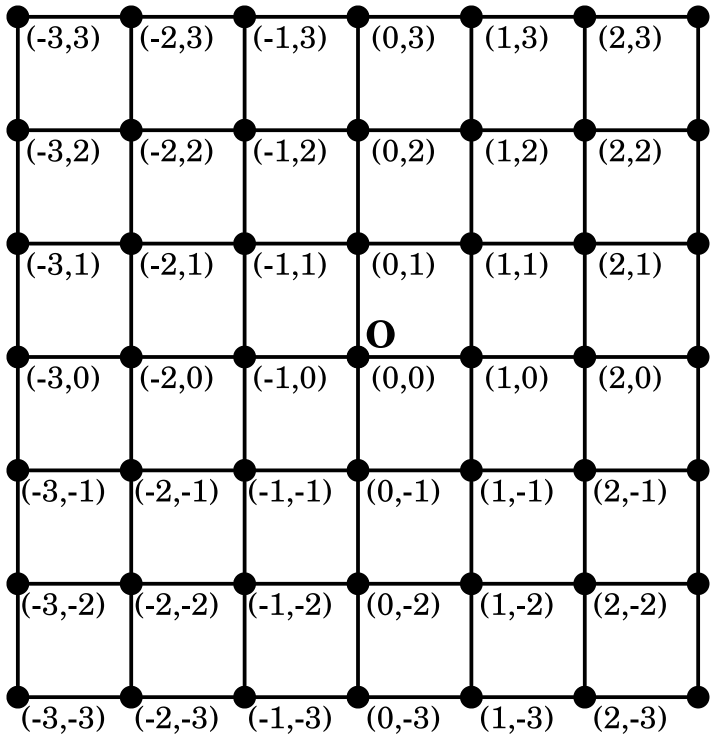

Figure 3.2: A Subset of $\mathbb{L}^2$

We now describe N-periodic percolation model. This model has $N+1$ parameters $(p_1,...,p_N.p_v) \in [0,1]^{N+1}$. All vertical edges, i.e., edges of the form $\langle (x,y),(x,y+1)\rangle$ for $x,y \in \mathbb{Z}$, have the property of being open with probability p_v. For horizontal edges, the edge $\langle u,v\rangle$ on $\mathbb{L}^2$ has coordinates (x,y) for u and coordinates $(x+1,y)$ for v, for some $x,y \in \mathbb{Z}^2$. The associated probabilities of each horizontal edge with coordinates $\langle (x,y),(x+1,y)\rangle$ has property of being open with probability p_{i+1}, where $i \equiv (x-y) \bmod N$. We will use the notation $\Theta_{\mathbb{L}^2}((p_1,p_2,...,p_N,p_v))$, where $(p_1,p_2,...,p_N,p_v) \in [0,1]^{N+1}$, to be the percolation probability of **O** on $\mathbb{L}^2$ using the N-periodic model. The *critical surface* in this chapter will be defined as the the set of $(N+1)$-tuples in $[0,1]^{N+1}$ that form the boundary between the sets of parameters where $\Theta_{\mathbb{L}^2}((p_1,p_2,...,p_N,p_v)) = 0$ and $\Theta_{\mathbb{L}^2}((p_1,p_2,...,p_N,p_v)) > 0$. Homogeneous models are obtained by restricting the parameters $(p_1,p_2,...,p_N,p_v)$ to the line segment in $[0,1]^{N+1}$ given by $\{(p_1,...,p_N,p_v) : p_1 = ... = p_N = p_v = p \text{ for } p \in [0,1]\}$. We call the intersection of this line segment with the critical surface the *critical percolation probability* $p_c(\mathbb{L}^2)$ and note that if $p < p_c(\mathbb{L}^2)$, then $\Theta_{\mathbb{L}^2}((p,...,p,p)) = 0$, while $\Theta_{\mathbb{L}^2}((p,...,p,p)) > 0$ for $p > p_c(\mathbb{L}^2)$. We remark that $p_c(\mathbb{L}^2) = 1/2$ (See Appendix C). More generally, when discussing a homogeneous percolation model on some other lattice **X**, we let p_c**(X)** denote the corresponding critical percolation probability.

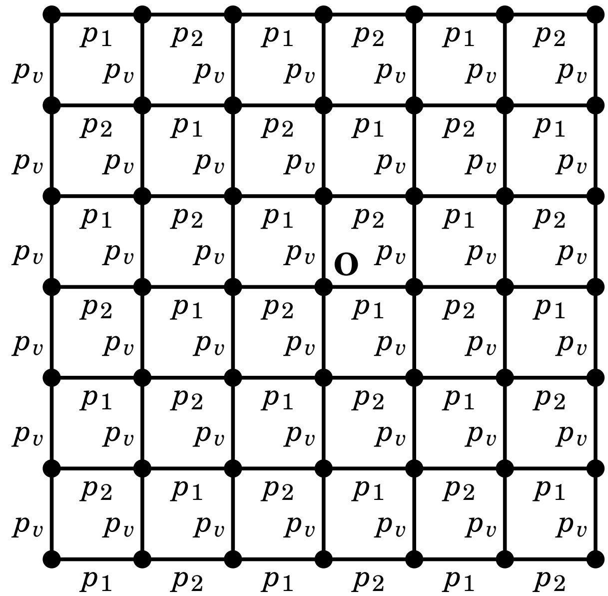

Figure 3.3: $\mathbb{L}^2$, 2-periodic bond percolation p_1, p_2, p_v

3.3 2-Periodic Bond Percolation on $\mathbb{L}^2$

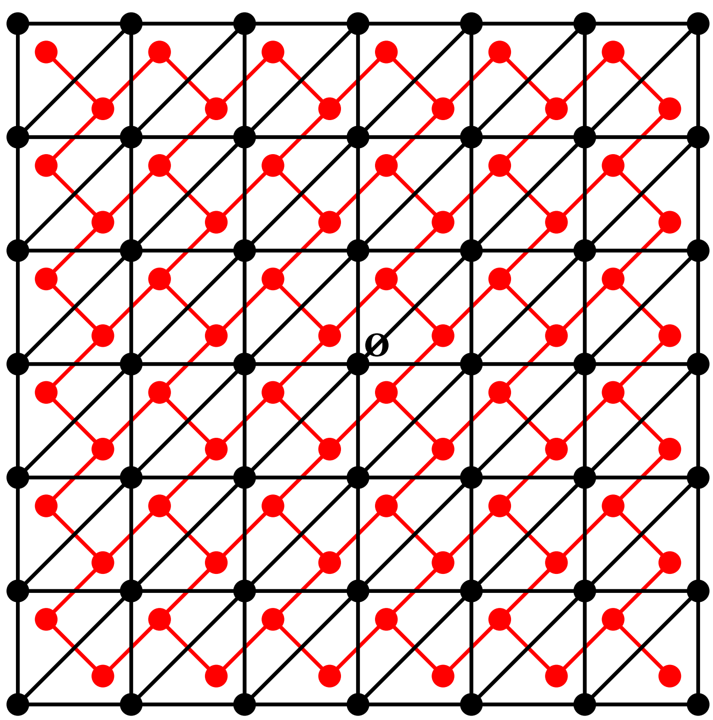

Figure 3.4: The triangular lattice in black with its dual in red.

In the examples considered in this section, two additional lattices arise. We begin by defining these here. The first is the triangular lattice $\mathbb{T}$. The triangular lattice $\mathbb{T}$ can be formed by starting with $\mathbb{L}^2$ and augmenting the edge set to become $\mathbb{E}(\mathbb{T}) = \mathbb{E}(\mathbb{L}^2) \cup \mathbb{D}$ where $\mathbb{D}$ is the set of 'diagonal' edges, $\mathbb{D} = \{\langle (x,y),(x+1,y+1)\rangle \mid x,y \in \mathbb{Z}\}$. Figure 3.4 shows the triangular lattice with black dots and line segments, along with its dual in red. The lattice $\mathbb{T}$ gets its name from the fact that the faces are all triangles.

We consider an inhomogeneous case, where the probabilities of the horizontal, vertical, and diagonal edges being open are given by p_h, p_v, and p_d, respectively. These parameters are distinct from those specified in Section 3.2. We use the notation $\Theta_{\mathbb{T}}(\mathbf{p})$ to denote the probability that the origin percolates on the triangular lattice. It is positive precisely when $p_h + p_v + p_d - p_h p_v p_d > 1$ or at least one of the parameters is equal to 1 [2, Theorem 11.116, Appendix C]. In particular, the critical surface is given by

$$p_h + p_v + p_d - p_h p_v p_d = 1. \tag{3.1}$$

As explained in Section 3.1, the dual of $\mathbb{T}$ is formed by placing a vertex in each face of $\mathbb{T}$ and edges between two vertices in the dual whose faces in $\mathbb{T}$ share an edge in $\mathbb{T}$. The dual of $\mathbb{T}$ is called the hexagonal lattice $\mathbb{H}$ because the faces of this graph are hexagons. This is illustrated in Figure 3.4. The probability of an edge in $\mathbb{T}$ that is the boundary of two faces of $\mathbb{T}$ being open determines the probability of the dual edge with endpoints that correspond to the faces of $\mathbb{T}$ being open. The dual edge that crosses the horizontal edge will be open with probability $1 - p_h$, the dual edge that crosses a vertical edge will be open with probability $1 - p_v$ and the diagonal edge that crosses a diagonal edge will be open with probability $1 - p_d$. It is useful to think of the realizations of configurations on the lattice and its dual as being generated jointly according to the rule that an edge in one lattice is open if and only if the corresponding edge in the other lattice is closed. In the examples analyzed later in this chapter, we will find it useful to know the critical surfaces for percolation on the triangular and hexagonal lattices. A description of these surfaces can be found in Appendix C. In addition, since $\mathbb{H}$ is the dual of $\mathbb{T}$, it follows that $\mathbb{T}$ is the dual of $\mathbb{H}$.

3.3.1 Example 1 ($p_2 = 1$)

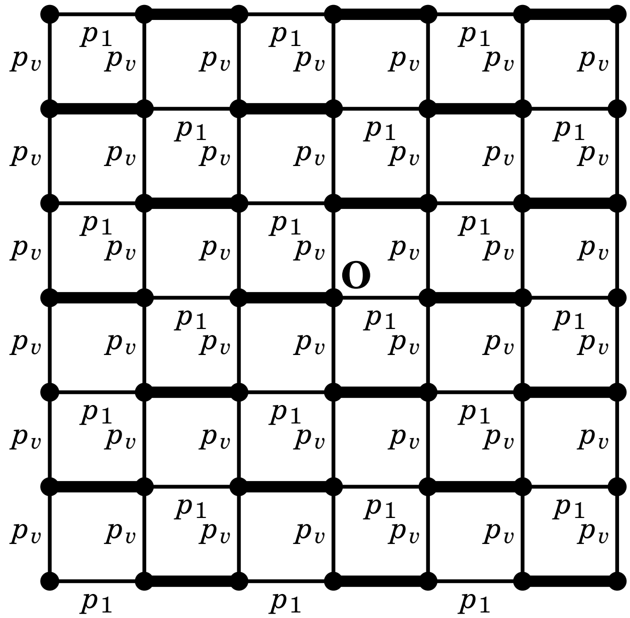

Figure 3.5: $\mathbb{L}^2$, where $p_2 = 1$

The first example is of 2-periodic percolation on $\mathbb{L}^2$ where $p_2 = 1$. We let each edge of the form $\langle(x,y),(x+1,y)\rangle$, $x,y \in \mathbb{Z}$, x,y even or $\langle(x+1,y),(x+2,y)\rangle$, x even and y odd, be open with probability $0 \le p_1 \le 1$ and all other horizontal edges be open with probability $p_2 = 1$. Figure 3.5 shows the graph, but the edges associated with the probability p_2 are drawn with heavier shading to show that they are open with probability 1, i.e., the edges that are emphasized. We may think of each edge that is on as a vertex. By "collapsing" these edges, we can redraw the lattice and see that it is equivalent to a certain triangular lattice (See Figure 3.6). Hence, we are considering bond percolation on $\mathbb{T}$ with $\mathbf{p} = (p_1, p_v, p_v)$ as a special case of 2-periodic bond percolation on $\mathbb{L}^2$ with $\mathbf{p} = (p_1, p_2, p_v) = (p_1, 1, p_v)$. Thus, $\Theta_{\mathbb{L}^2}((p_1,1,p_v)) = \Theta_{\mathbb{T}}((p_1,p_v,p_v))$.

Note that if $p_1 = 1$ we have percolation along the x-axis regardless of how many vertical bonds may be open or closed, i.e., for all $0 \le p_v \le 1$. In addition, if $p_v = 1$, percolation occurs for all $0 \le p_1 \le 1$. If $p_v = 0$, then there is percolation if and only if $p_1 = 1$. Note, if $p_1 = 0$, deleting the horizontal bonds with probability p_1, we then have the resulting graph that can be thought of as the square lattice. We recognize the resulting lattice as the squre lattice rotated by 45°. Furthermore, the percolation model is now homogeneous with parameter p_v and we know that the critical percolation probability is $p_c(\mathbb{L}^2) = \dfrac{1}{2}$ (See Appendix C). Therefore, percolation probability is positive when $p_1 = 0$ and $p_v > \frac{1}{2}$. In the case where $0 < p_v = p_1$, then we have just the triangular lattice and the critical probability would be $p_c(\mathbb{T}) = 2\sin(\pi/18)$ (See Appendix C).

Next we consider the case $0 < p_1 < 1$ and $0 < p_v < 1$. By (3.1), the percolation probability of the triangular lattice, $\Theta_{\mathbb{T}}(\mathbf{p})$ is positive if $p_h + p_v + p_d - p_h p_v p_d > 1$. In our case two parameters are equal to p_v while the other is p_1. Thus, we have the condition $2p_v + p_1 - p_1 p_v^2 > 1$. Therefore, when $p_2 = 1$, the critical surface is given by the equation $2p_v + p_1 - p_1 p_v^2 = 1$. If we isolate p_1 in our previous inequality, we get that $p_1 > \dfrac{1-2p_v}{1-p_v^2}$, since $p_v < 1$. This means that, as long as p_1 is greater than $\dfrac{1-2p_v}{1-p_v^2}$, we will have percolation. If we solve for p_v in the equation $-p_1 p_v^2 + 2p_v + p_1 - 1 = 0$, we get $p_v = \dfrac{1 \pm \sqrt{1+p_1(p_1-1)}}{p_1} = \dfrac{1 \pm \sqrt{1-p_1(1-p_1)}}{p_1}$. Since $0 \le p_1(1-p_1) \le \dfrac{1}{4}$ on $[0,1]$, we have $1 - p_1(1-p_1) > 0$. We can reject the solution that involves adding the radical since, when $0 < p_1 < 1$,

$$\frac{1+\sqrt{1+p_1(p_1-1)}}{p_1} > 1 + \sqrt{1+p_1(p_1-1)} > 1, \tag{3.2}$$

and p_v cannot be greater than 1. Then we must subtract the radical. So, if $p_v > \dfrac{1-\sqrt{1+p_1(p_1-1)}}{p_1}$, then percolation will occur. The graph of $\dfrac{1-\sqrt{1+p_1(p_1-1)}}{p_1}$ is plotted in Figure 3.7.

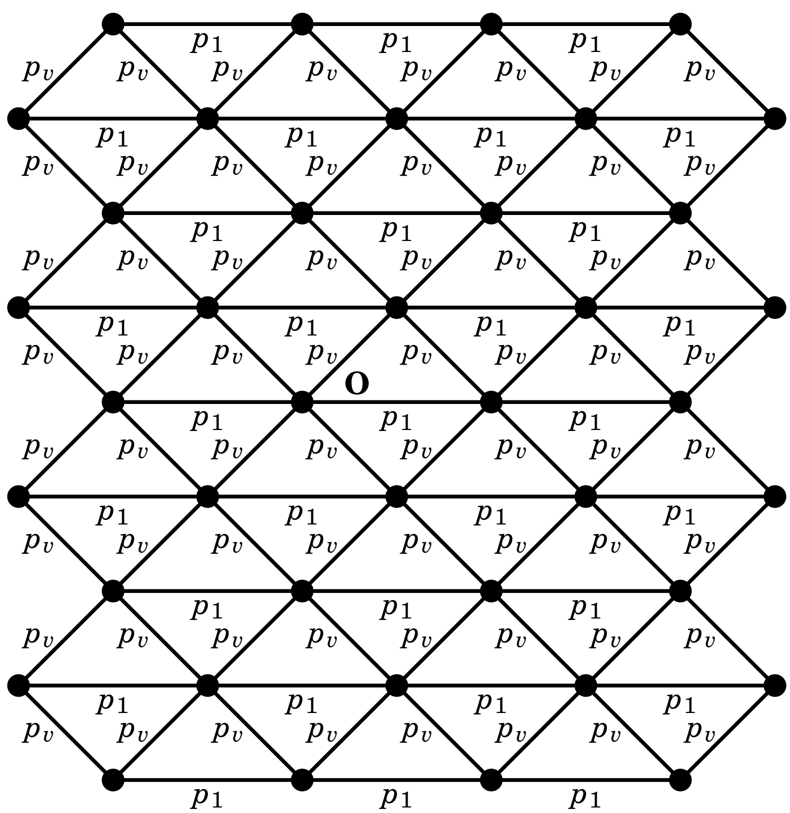

Figure 3.6: $\mathbb{L}^2$, where edges that are turned on are "collapsed".

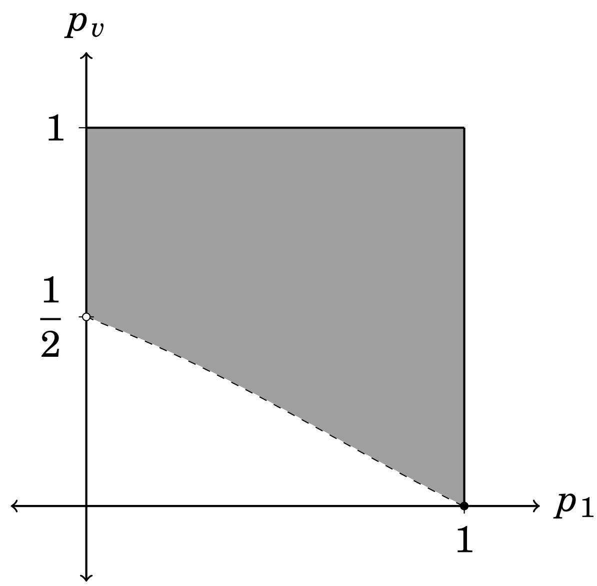

Figure 3.7: Region where **O** percolates for Example 1 ($p_2 = 1$)

3.3.2 Example 2 ($p_2 = 0$)

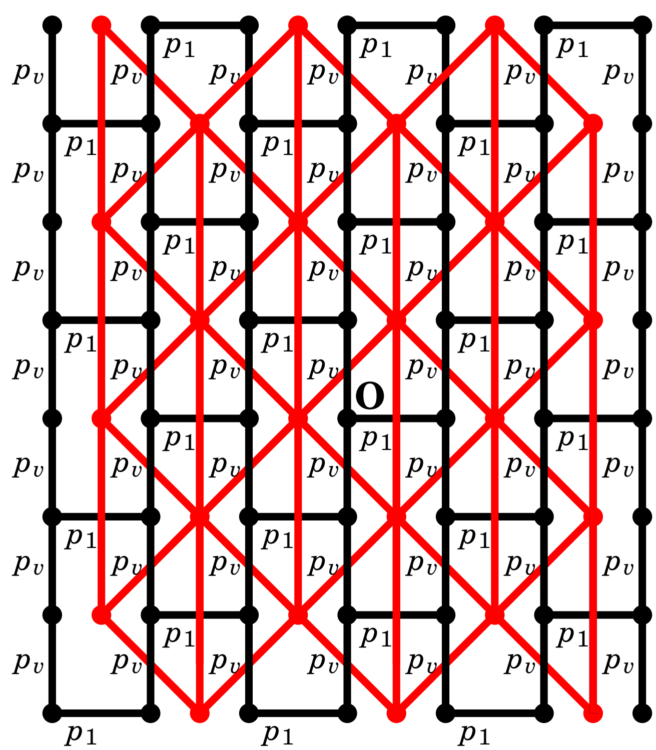

Figure 3.8: $\mathbb{L}^2$, where p_1, $p_2 = 0$ and the "dual" lattice

In this subsection, we examine another of the simplest nontrivial examples of 2-periodic bond percolation on the square lattice, the case where $p_2 = 0$. It is illustrated in black in Figure 3.8 after the edges that have probability 0 of being open are deleted. Recall that in the general case of 2-periodic bond percolation there are three parameters p_1, p_2 and p_v but here we have specialize by setting $p_2 = 0$. That is, all vertical edges are open with probability $0 < p_v \leq 1$ and horizontal edges are either closed with probability one (because $p_2 = 0$) or open with probability p_1. Figure 3.8 demonstrates that once all closed edges are deleted from $\mathbb{L}^2$, the resulting graph is the hexagonal lattice $\mathbb{H}$, shown in black along with the dual graph $\mathbb{T}$ in red. So we are effectively considering bond percolation on $\mathbb{H}$ with parameters (p_1, p_v, p_v) as a special case of 2-periodic bond percolation on the square lattice in which $p_2 = 0$, i.e. the parameters for $\mathbb{L}^2$ are $(p_1, 0, p_v)$. In particular, $\Theta_{\mathbb{L}^2}(p_1, 0, p_v) = \Theta_{\mathbb{H}}(p_1, p_v, p_v)$, where $\Theta_{\mathbb{H}}(\mathbf{p})$ is the probability that the origin percolates in $\mathbb{H}$ with parameters $\mathbf{p}$.

We begin by examining the special case where $p_1 = p_v$. In this situation, we have a homogeneous percolation model on a hexagonal lattice. The critical point for this model, $p_c(\mathbb{H}) \in [0,1]$, $[\Theta_{\mathbb{H}}((p,p,p)) > 0$ for $p > p_c(\mathbb{H})$ and $\Theta_{\mathbb{H}}((p,p,p)) = 0$ for $p < p_c(\mathbb{H})]$ is related to the critical point of the dual triangular model, $p_c(\mathbb{T}) \in [0,1]$, $[\Theta_{\mathbb{T}}((p,p,p)) > 0$ for $p > p_c(\mathbb{T})$ and $\Theta_{\mathbb{T}}((p,p,p)) = 0$ for $p < p_c(\mathbb{T})]$ by $p_c(\mathbb{H}) + p_c(\mathbb{T}) = 1$ (See Appendix C). Additionally note that $p_c(\mathbb{T}) = 2\sin(\pi/18)$ (See Appendix C). So $p_c(\mathbb{H}) = 1 - 2\sin(\pi/18)$. Thus, the point $(1-2\sin(\pi/18), 0, 1-2\sin(\pi/18))$ must lie on the critical surface for the 2-periodic percolation model with $p_2 = 0$.

More generally, the critical surface for percolation on the hexagonal lattice with parameters (p_1, p_v, p_v) is the same as the critical surface for percolation on the triangular lattice with parameters $(1-p_1, 1-p_v, 1-p_v)$ (See Appendix C). From (3.1), we have that

$$(1-p_1)+(1-p_v)+(1-p_v)-(1-p_1)(1-p_v)(1-p_v) = 1.$$

We rewrite this equation as follows:

$$\begin{aligned} 1 =& (1-p_1)+(1-p_v)+(1-p_v)-(1-p_1)(1-p_v)(1-p_v) \\ =& 3 - p_1 - 2p_v - (1 - 2p_v + p_v^2 - p_1 + 2p_1p_v - p_1p_v^2) \\ =& 2 - p_v^2 - 2p_1p_v + p_1p_v^2. \end{aligned} \tag{3.3}$$

Thus,

$$0 = (p_1 - 1)p_v^2 - 2p_1p_v + 1. \tag{3.4}$$

Before proceeding further, let's examine two special cases, $p_1 = 0$ and $p_1 = 1$. First, if we let $p_1 = 0$, i.e., the case where all horizontal edges are closed in our newly formed hexagonal lattice, then we have $-p_v^2 + 1 = 0$. The only reasonable solution is $p_v = 1$, which makes $\Theta_{\mathbb{L}^2}(0,0,1) = \Theta_{\mathbb{H}}(0,1,1) = 1$. In this situation there is percolation along the vertical line that passes through **O**. Second, if $p_1 = 1$, then $-2p_v + 1 = 0$, then the solution is $p_v = 1/2$. Thus $\Theta_{\mathbb{L}^2}(1,0,p_v) = \Theta_{\mathbb{H}}(1,p_v,p_v) > 0$ if $p_v > 1/2$. The resulting graph can be thought as a regular square lattice where the critical value is $p_c(\mathbb{L}^2) = 1/2$, which is already known [2, Theorem 11.11, Appendix C]. To recognize this as a "regular square lattice" observe that every bond that is on (open with probability 1) can be redrawn, where we "collapse" each edge that is on to a single point and think of each of these as a vertex. Only the vertical edges would be drawn if we also omit the edges that are off (open with probability zero).

In the more general situation, we have $0 < p_1 < 1$. The condition for the critical surface is given in (3.4). For each $p_1 \in (0,1)$, we will find a critical value $p_{v,c}$ for which the model with $p_1 \in (0,1)$ fixed, $p_2 = 0$, and $p_{v,c}$ is on the critical surface in $[0,1]^3$. Indeed, this critical value $p_{v,c}$ will be a function of p_1. We begin by using the quadratic formula to write,

$$p_{v,c} = \frac{-(-2p_1) \pm \sqrt{(-2p_1)^2 - 4(p_1-1)(1)}}{2(p_1-1)} = \frac{2p_1 \pm \sqrt{4p_1^2 - 4(p_1-1)}}{2(p_1-1)} = \frac{p_1 \pm \sqrt{1-p_1+p_1^2}}{p_1-1}.$$

Because $0 < p_1 < 1$, we have $1-p_1 > 0$ and so the discriminant $1-p_1+p_1^2$ is positive. Due to the denominator, we must reject the root $\dfrac{p_1 + \sqrt{1-p_1+p_1^2}}{p_1-1}$ as the root is negative. Since $p_1 - \sqrt{1-p_1+p_1^2} < p_1 - \sqrt{p_1^2} = 0$, the other root is positive. Also, since $1-p_1+p_1^2 = 1-p_1(1-p_1) < 1$ for $0 < p_1 < 1$, we have that $0 < \dfrac{p_1 - \sqrt{1-p_1+p_1^2}}{p_1-1} = \dfrac{\sqrt{1-p_1+p_1^2} - p_1}{1-p_1} < \dfrac{1-p_1}{1-p_1} < 1$. Thus, the other root actually lies in the interval $(0,1)$. So if we take $p_{v,c} = \dfrac{p_1 - \sqrt{1-p_1+p_1^2}}{p_1-1}$, then for $p_v > p_{v,c}$ we have $\Theta_{\mathbb{L}^2}(p_1,0,p_v) = \Theta_{\mathbb{H}}(p_1,p_v,p_v) > 0$ and for $p_v < p_{v,c}$ we have $\Theta_{\mathbb{L}^2}(p_1,0,p_v) = \Theta_{\mathbb{H}}(p_1,p_v,p_v) = 0$. Figure 3.9 shows the region of ordered parameter (p_1,p_v) pairs where bond percolation occurs for 2-periodic bond percolation on $\mathbb{L}^2$ with $p_2 = 0$. Please note that, although the boundary curve may appear to be linear at a quick glance, it is not the graph of a straight line. Notice when using Taylor approximation at $p_1 = 0$ the curve of $\dfrac{\sqrt{1-p_1+p_1^2} - p_1}{1-p_1}$ it is approximately $1 - \dfrac{1}{2}p_1$, which is linear.

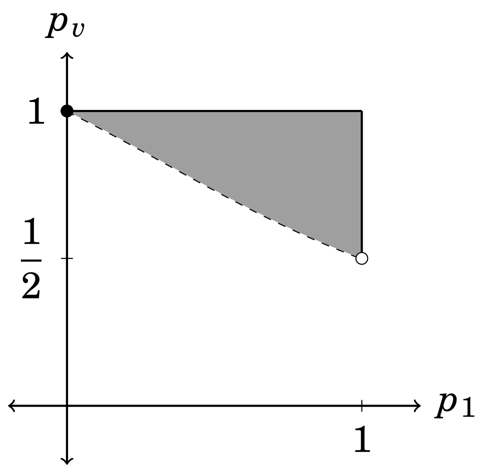

Figure 3.9: Region where percolation on the origin O for Example 1

Since our formula $p_{v,c} = \dfrac{\sqrt{1-p_1+p_1^2}-p_1}{1-p_1}$ cannot be evaluated at $p_1 = 1$, we use L'Hopitals Rule to show that

$$\lim_{p_1 \uparrow 1} \frac{\sqrt{1-p_1+p_1^2}-p_1}{1-p_1} = \lim_{p_1 \uparrow 1} \frac{\dfrac{-1+2p_1}{2\sqrt{1-p_1+p_1^2}}-1}{-1} = \frac{\dfrac{1}{2}-1}{-1} = \frac{1}{2},$$

which is consistent with what we found when we examined the special case where $p_1 = 1$. Likewise, if $\lim_{p_1 \downarrow 0} \dfrac{\sqrt{1-p_1+p_1^2}-p_1}{1-p_1} = 1$, which in this case would imply $p_v = 1$.

3.3.3 Summary of 2-Periodic Percolation on $\mathbb{L}^2$

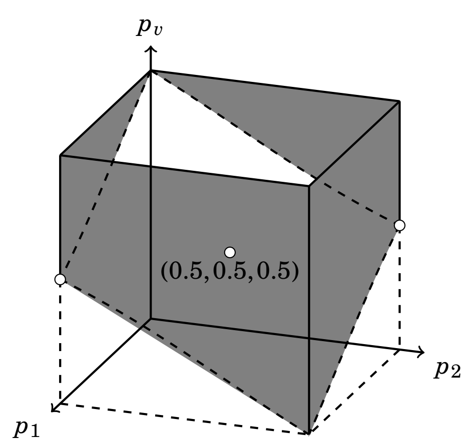

Figure 3.10: 3-D representation of regions of positive percolation probability

With Figure 3.10, we visually summarize our results of the regions in gray where percolation probability is positive.

3.4 3-Periodic Bond Percolation on $\mathbb{L}^2$

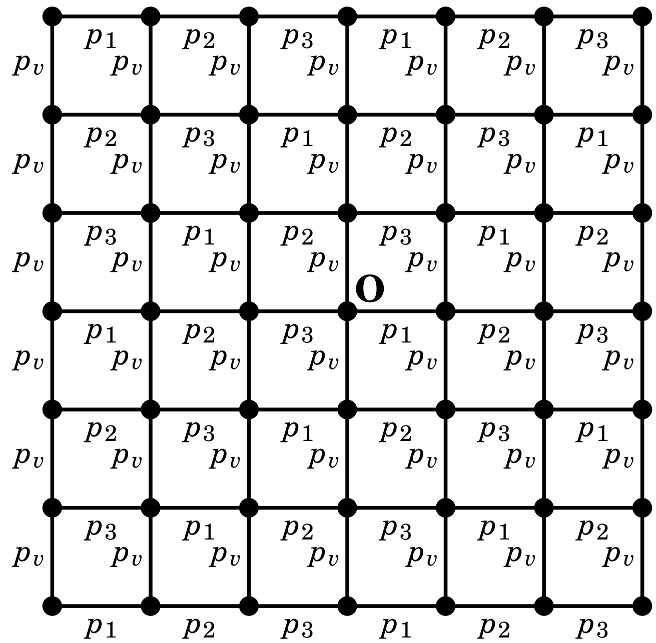

Figure 3.11: $\mathbb{L}^2$, 3-periodic bond percolation p_1, p_2, p_3, p_v

For 3-periodic bond percolation, we start with the $\mathbb{L}^2$ lattice. Every edge of the form $\langle(x,y),(x,y+1)\rangle$ where $x,y \in \mathbb{Z}$ will be open with probability $0 \le p_v \le 1$. The following description of the probabilities of the horizontal edges partitions the edges and is equivalent to the definition given in section 3.2 so as to emphasize the periodicity of the probabilities of the edges. Edges of the form $\langle(x,y),(x+1,y)\rangle$ are horizontal edges. Every edge of the form $\langle(x,y),(x+1,y)\rangle$ such that $y \equiv 0 \pmod 3$ shall have an associated probability such that if $x \equiv 0 \pmod 3$, the probability that it is open is p_1, if $x \equiv 1 \pmod 3$, the probability that it is open is p_2, if $x \equiv 2 \pmod 3$, the probability that it is open is p_3. Every edge of the form $\langle(x,y),(x+1,y)\rangle$ such that $y \equiv 1 \pmod 3$ shall have an associated probability such that if $x \equiv 1 \pmod 3$, the probability that it is open is p_1, if $x \equiv 2 \pmod 3$, the probability that it is open is p_2, if $x \equiv 0 \pmod 3$, the probability that it is open is p_3. Every edge of the form $\langle(x,y),(x+1,y)\rangle$ such that $y \equiv 2 \pmod 3$ shall have an associated probability such that if $x \equiv 2 \pmod 3$, the probability that it is open is p_1, if $x \equiv 0 \pmod 3$, the probability that it is open is p_2, if $x \equiv 1 \pmod 3$, the probability that it is open is p_3.

Some trivial cases are as follows. If $p_1 = p_2 = p_3 = 1$, then percolation will occur. If $p_v = 0$ and if for some $i = 1,2,3$, $p_i < 1$, then percolation will not occur. Also, if $p_i = 0$ for $i = 1,2,3$ and $p_v < 1$, percolation will also not occur.

3.4.1 Example 3

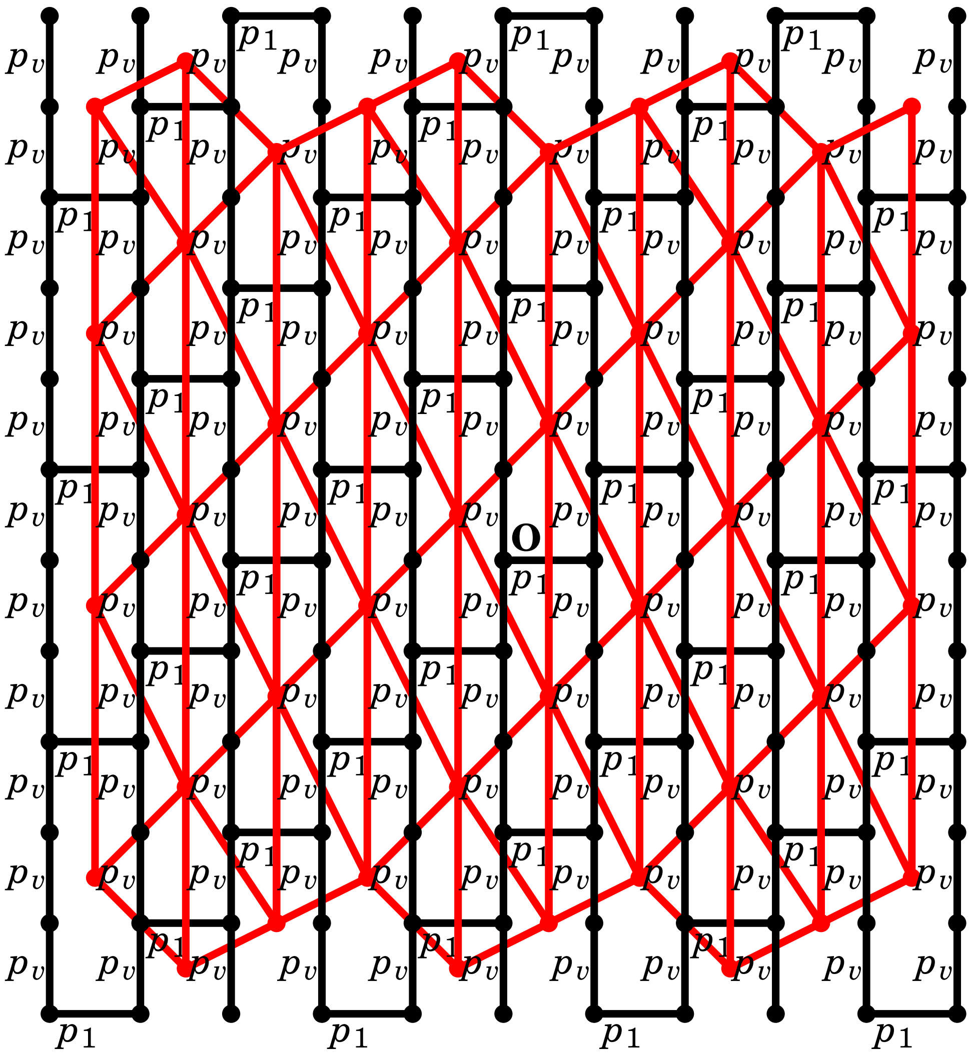

Figure 3.12: $\mathbb{L}^2$, 3-periodic, where two horizontal edges are turned off.

Figure 3.12 shows the scenario where $p_2 = p_3 = 0$ while p_1 and p_v each vary over $(0,1)$. Note that the graph can be thought as a hexagonal lattice since the dual graph is the triangular lattice. One pair of sides of each hexagon is formed by two vertical bonds. The probability of both bonds being open is p_v^2. Another pair of sides is formed by another pair of vertical bonds. Each is open with probability p_v. The last pair of sides is formed by the horizontal bonds, which are open with probability p_1. In particular, $\Theta_{\mathbb{L}^2}(p_1,0,0,p_v) = \Theta_{\mathbb{H}}(p_1,p_v^2,p_v)$. The dual graph $\mathbb{T}$ of $\mathbb{H}$ has bonds with associated probability $1-p_v^2$ for the edge that have endpoints on the pair of faces that share the boundary with two vertical edges. The bond in the dual the has endpoints on face that share the boundary with the horizontal edge that has probability p_1, will have associated probability $1-p_1$. The third pair of sides of the dual graph will have probability $1-p_v$. In particular, $\Theta_{\mathbb{L}^2}(p_1,0,0,p_v) = \Theta_{\mathbb{H}}(p_1,p_v^2,p_v) = \Theta_{\mathbb{T}}(1-p_1,1-p_v^2,1-p_v)$.

In the case where $p_1 = 1$, the graph becomes the square lattice, with one pair of opposite sides open with probability p_v and the other side with two bonds, this side is open with probability p_v^2. The condition for the critical surface of the square lattice is $p_h + p_v = 1$ where p_h is the parameter for the probability of the "horizontal" edge to be open and p_v the parameter for the probability of the "vertical" edge to be open [2, Theorem 11.115, Appendix C]. We can let $p_h = p_v^2$. Substituting the values for the equation for the critical surface results in $p_v^2 + p_v = 1$. Solving for p_v using the quadratic formula, we get $p_v = \dfrac{-1 \pm \sqrt{5}}{2}$. Since subtracting a $\sqrt{5}$ results in a negative value, we only accept the solution $p_v = \dfrac{\sqrt{5}-1}{2}$ which is the critical value in this particular scenario.

3.4.2 Example 4

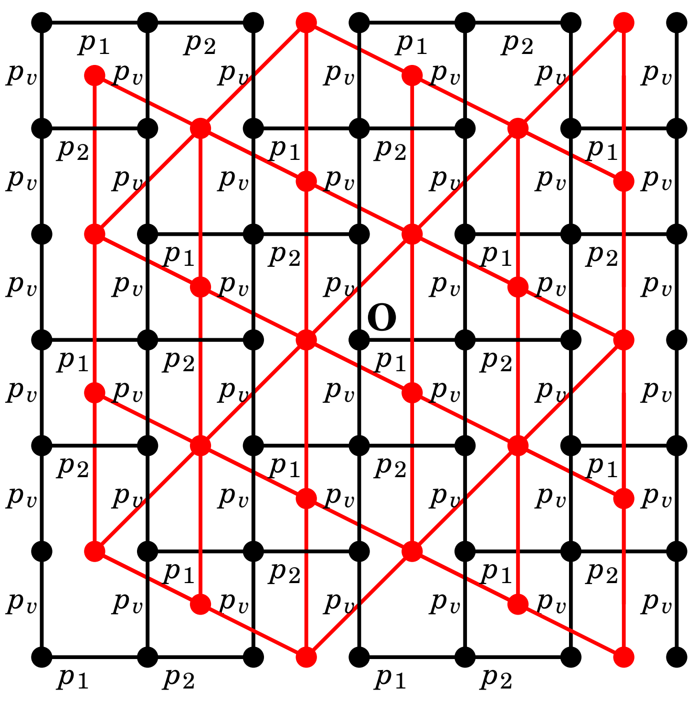

Figure 3.13: $\mathbb{L}^2$, 3-periodic, where one horizontal edge is turned off.

In this example we let $p_3 = 0$, see figure 3.13. The resulting graph can be thought as composed of faces whose boundaries are either a square and a hexagonal. The conditions for the critical surface are not know for the graph, but if we notice the dual of the graph, we see that it is a known lattice called the "bow-tie" lattice $\mathbb{B}$. If $p_v = p_1 = p_2 = p$, then the critical point is known to be the solution of the polynomial $1-p-6p^2+6p^3-p^5=0$ on $(0,1)$ [5, pg. 1528]. The critical point is $p_c(\mathbb{B}) \approx 0.404518$. This implies that the graph formed by setting $p_3 = 0$ and letting the parameters take on the same probability, then the critical value of this 3-periodic bond percolation on $\mathbb{L}^2$ with $p_3 = 0$ and $p_1 = p_2 = p_v = 1-p$ is $1-p_c(\mathbb{B}) \approx 0.595482$. Unfortunately, for the inhomogeneous case where none of the parameters have the same probability, the critical surface is not easily determined because the star-triangle transformation (the method used to find the critical condition in the homogeneous case) cannot be used.

3.4.3 Example 5

The next example, we set $p_1 = 1$, $p_2 = 1$, $p_3 = 0$ and we let p_v vary. Looking at Figure 3.13, the horizontal edges that are on, along with their endpoints can be thought as one vertex. The resulting lattice when the edges that are on are condensed into a vertex will have a lattice that looks like Figure 3.14. Note, there are two edges that are diagonal for every pair of vertices of the form (x,y) and $(x+1,y+1)$. The resulting graph can be treated as a square lattice. The vertical edge has p_v as the probability of being open. Each "horizontal edge," referring to the pair of diagonal edges has the probability of $1-(1-p_v)^2$ of being open since the only way this "edge" can be closed is if both edges are closed. From [2, Appendix C], we know that the critical surface is given by $p_h + p_v = 1$ where p_h the probability that the horizontal edge is open and p_v the probability that the vertical edge is open. Thus,

$$1 = (1-(1-p_v)^2) + (p_v) \tag{3.5}$$

$$0 = p_v - (1-p_v)^2 = p_v - (1-2p_v+p_v^2) = -p_v^2 + 3p_v - 1 \tag{3.6}$$

or

$$p_v^2 - 3p_v + 1 = 0 \tag{3.7}$$

When we apply the quadratic formula, we get $p_v = \dfrac{3 \pm \sqrt{(3)^2 - 4(1)(1)}}{2(1)}$. Since adding the radical will give a value larger than 1, we only take $p_v = \dfrac{3-\sqrt{5}}{2}$ which is approximately 0.38197.

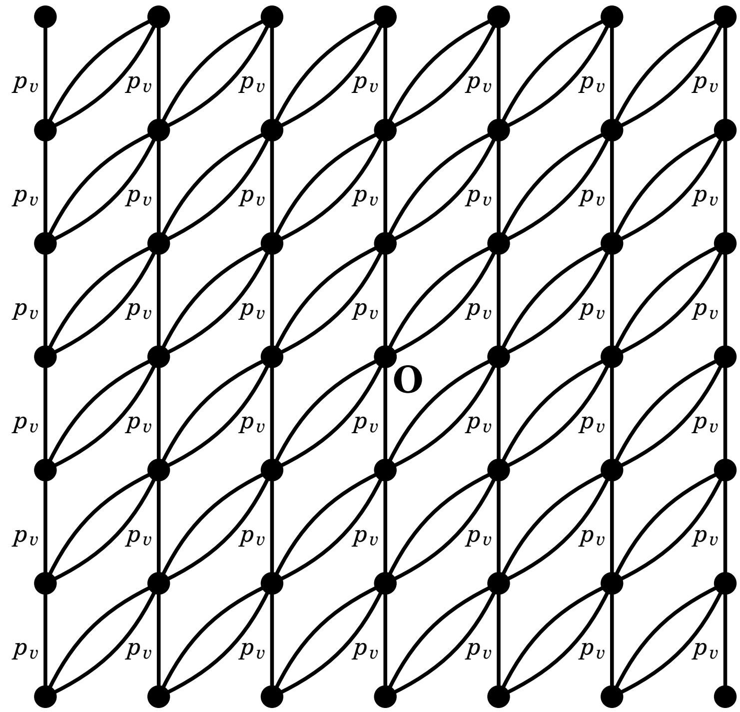

Figure 3.14: $\mathbb{L}^2$, 3-periodic, where two horizontal edges are turned on and the third is off.

3.4.4 Example 6

In this example, we consider the situation where one set of horizontal bonds are on, another set is off and a third set of bonds vary, without loss of generality, we suppose p_1 varies, $p_2 = 1$ and $p_3 = 0$. If we "collapse" the bonds that are on and think of each collapsed edge as a vertex and delete the bonds that are off the resulting lattice is as Figure 3.15. The lattice is know as the martini-B lattice. For the homogeneous case where $p_1 = p_v = p$ the critical value for the martini-B lattice is given by the solution to the equation $(2p-1)(p^2-p-1) = 0$ [6, Section 2.3]. The solution that is in the interval $[0,1]$ is $p = \frac{1}{2}$. Note the dual of the martini-B lattice is also the martini-B lattice. In the case where the bonds with probability p_1 are off, i.e., if $p_1 = 0$ and we delete the bonds with probability p_1, the resulting lattice is the hexagonal lattice the critical value is $1-2\sin(\pi/18) \approx 0.6527$. Thus, if $p_v > 1-2\sin(\pi/18)$, there is a positive probability of percolation, $\Theta_{\mathbb{L}^2}(0,1,0,p_v) > 0$. If $p_1 = 1$ and we "collapse" the edges that are on, the resulting lattice is the same as in Example 5, thus the critical point is $p_v = \frac{3-\sqrt{5}}{2} \approx 0.38197$. Therefore, when $p_1 = 1$, $\Theta_{\mathbb{L}^2}(1,1,0,p_v) > 0$ when $p_v > \frac{3-\sqrt{5}}{2}$.

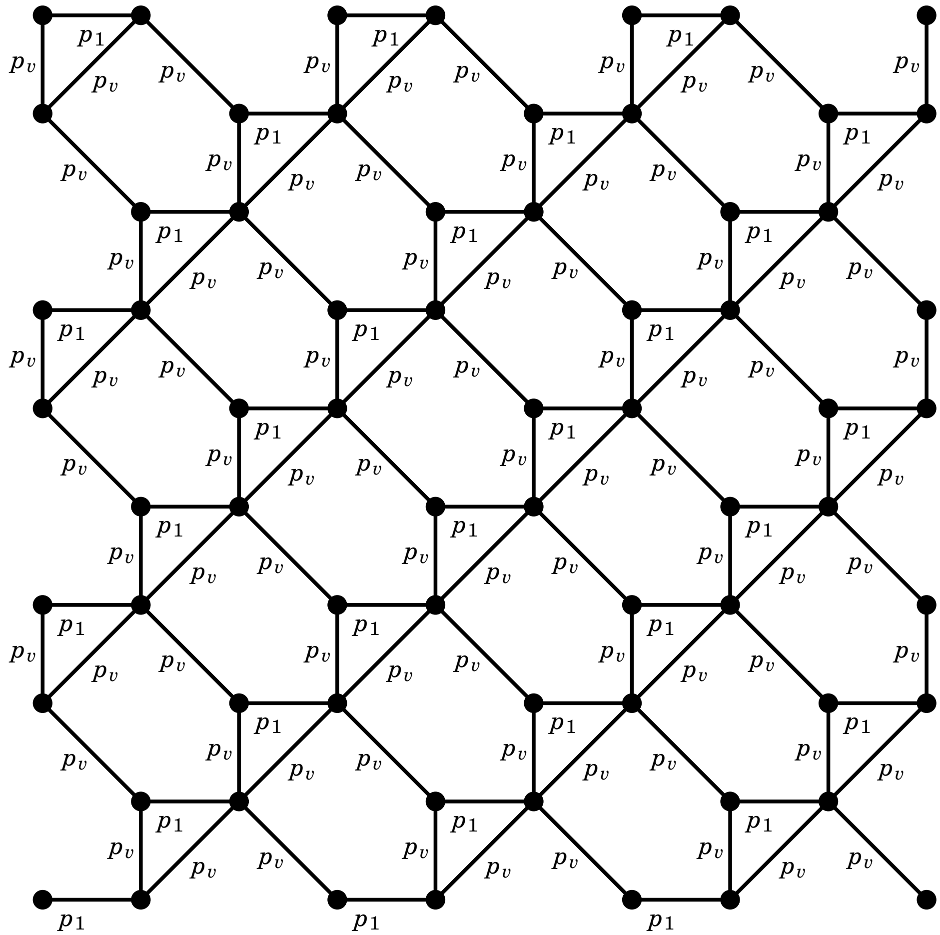

Figure 3.15: $\mathbb{L}^2$, 3-periodic, where one horizontal edge is on, another is off and the third is free.

3.4.5 Example 7

If we set $p_1 = 1$ and $p_2 = 1$ and we let p_3 and p_v vary, we can "collapse" the bonds that are on into vertices to form a new graph. The new graph new looks like Figure 3.16 where the two pair of diagonal bonds are open with probability p_v and the horizontal edge is open with probability p_3. We may use the equation for the critical surface for a triangular lattice, where $p_h = p_3$, $p_v = p_v$, and $p_d = 1-(1-p_v)^2$, to determine the critical surface in this case. Then $\mathbf{p} = (p_3, p_v, 1-(1-p_v)^2)$ is on the critical surface if and only if

$$p_3 + p_v + (1-(1-p_v)^2) - p_3 p_v (1-(1-p_v)^2) = 1. \tag{3.8}$$

We examine the extreme cases where $p_3 = 1$, $p_3 = 0$, $p_v = 0$, or $p_v = 1$, before looking at the more general situation where $0 < p_3 < 1$ and $0 < p_v < 1$. When $p_3 = 1$, there is percolation for all $0 \le p_v \le 1$. If $p_3 = 0$, we have the lattice of Figure 3.14 and equation (3.8) is the same as equation (3.6), thus $\Theta_{\mathbb{L}^2}(1,1,0,p_v) > 0$ when $p_v > \frac{3-\sqrt{5}}{2} \approx 0.38197$. If $p_v = 0$, then there is pecolation if and only if $p_3 = 1$. If $p_v = 1$, then there is percolation for all $0 \le p_3 \le 1$.

Next we consider $0 < p_3 < 1$ and $0 < p_v < 1$. If we solve for p_3 in (3.8), we get $p_3 = \frac{(1-p_v)^2 - p_v}{1-p_v(1-(1-p_v)^2)}$, so $\Theta_{\mathbb{L}^2}(1,1,p_3,p_v) > 0$ when $p_3 > \frac{(1-p_v)^2 - p_v}{1-p_v(1-(1-p_v)^2)}$.

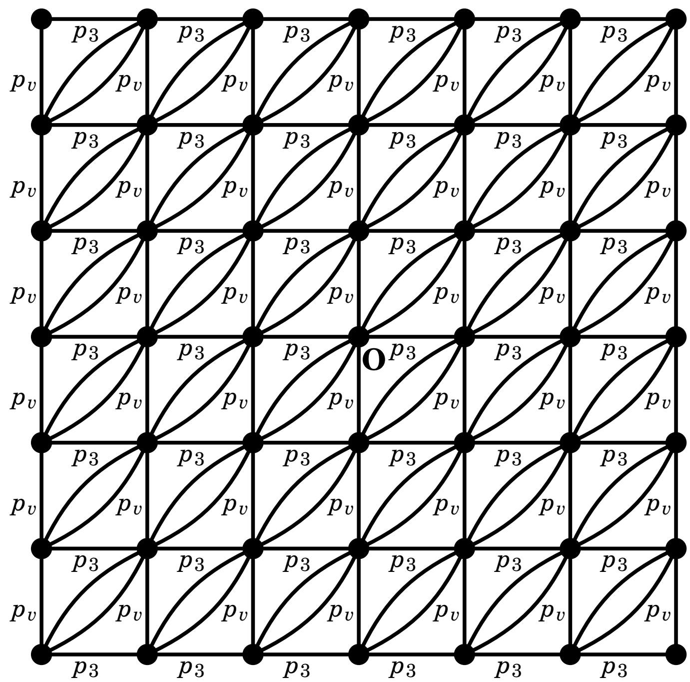

Figure 3.16: $\mathbb{L}^2$, 3-periodic, where two horizontal edges are turned on and the third varies.

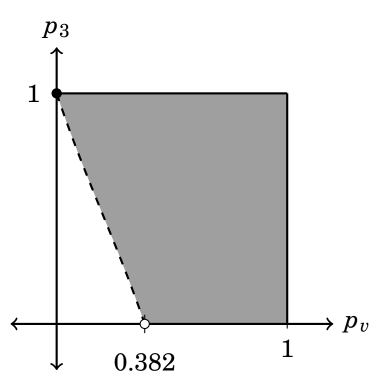

Figure 3.17: Region where $\Theta_{\mathbb{L}^2}(1,1,p_3,p_v) > 0$ where $p_3, p_v \in [0,1]$.

3.5 Generalizations

For arbitrary $N \in \mathbb{R}_+$ the critical surface for N-periodic bond percolation is not known. However, one case that does generalize is $p_1 > 0$ or $p_N > 0$, $p_v > 0$ and all other horizontal bonds are off. By deleting all bonds that are off, the resulting graph can be thought as a hexagonal lattice where one pair of sides of the "hexagonal" has probability of being open p_1, another pair of side are open with probability p_v, and the other pair of "sides" where each side has $N-1$ vertical bonds where each side is consider open with probability p_v^{N-1}. Figure 3.18 shows 5-periodic bond percolation where $p_1 = 1$ and $p_i = 0$ for all other i and p_v varies on $[0,1]$. In this case for general $N \in \mathbb{R}_+$, the critical surface is the set of parameter values $\mathrm{p} = (p_1, 0, ..., 0, p_v)$ that satisfy $(1-p_1)+(1-p_v)+(1-p_v^{N-1})-(1-p_1)(1-p_v)(1-p_v^{N-1}) = 1$.

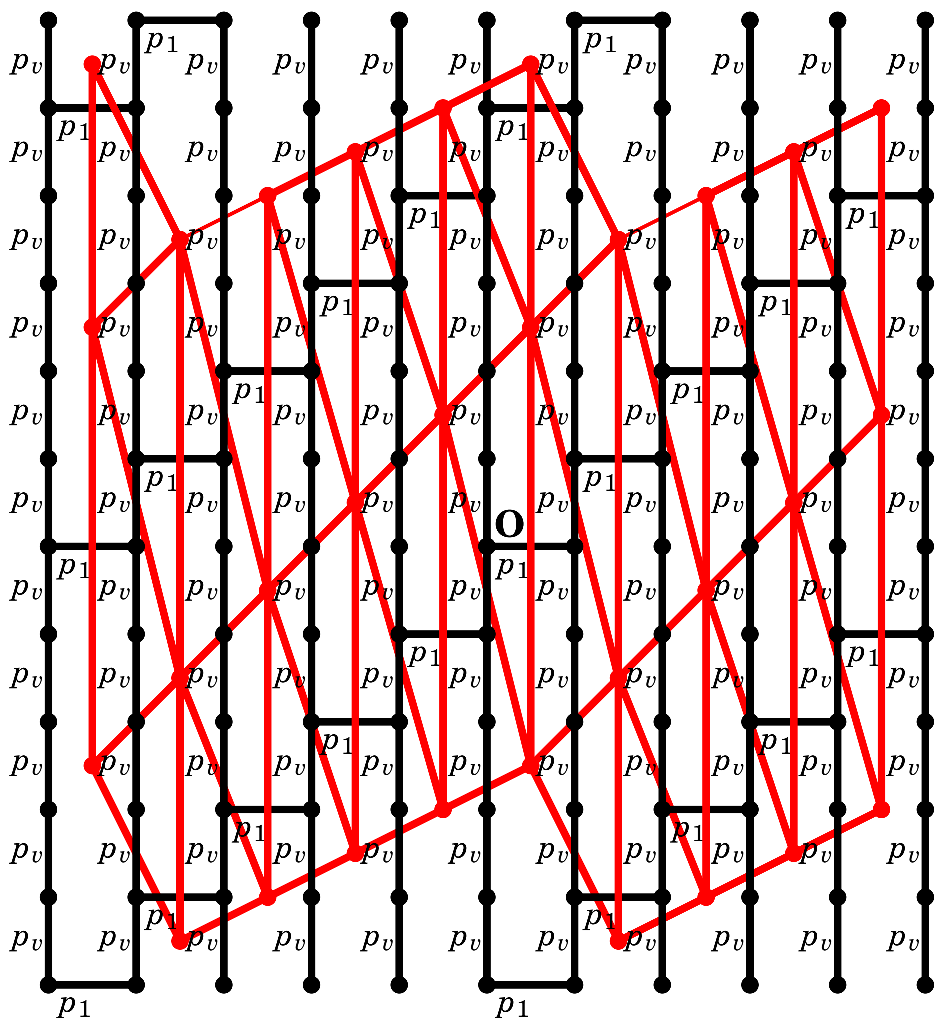

Figure 3.18: $\mathbb{L}^2$, 5-periodic, where all horizontal bonds are off except for bonds with associated probability p_1.

If 2 divides N, for some $N \in \mathbb{Z}_+$ and if $p_i = p_j$ for $i \equiv j \pmod 2$, $i,j \in \{1,2,...,N\}$, then this case reduces to the same cases of Section 3.3. Likewise, if 3 divides N, for some $N \in \mathbb{Z}_+$ and if $p_i = p_j$ for $i \equiv j \pmod 3$, $i,j \in \{1,2,...,N\}$ where this case is the same as in Section 3.4. The lattices with other for percolation that are more difficult to find.

3.6 Conclusion and Directions for Further Research

In this chapter, we have explored various cases of N-periodic percolation for $N = 2$ and $N = 3$ by restricting certain parameters values. This has lead to interesting insights concerning the critical surface. However, complete characterization of the critical surface for $N = 2$ or $N = 3$ remains as an open problem. The challenge for $N \geq 4$ is even greater because resulting lattices where we may "delete" or "collapse" bonds to form other lattices do not result in graphs with a known critical surface. Another direction for further exploration might be to allow vertical bonds to have distinct probabilities of being open, i.e., perhaps to allow periodic behavior, or some thing of this nature.

Appendices

Appendix A

Alternative Proof from Section 2.5

Here, we give another proof of

$$\prod_{i=1}^{r}(1-p_i\hat{\theta}_{r,\vec{p}}) = 1-\left(\sum_{i_1=1}^{r} p_{i_1}\right)\hat{\theta}_{r,\vec{p}} + \left(\sum_{1\le i_1<i_2\le r} p_{i_1}p_{i_2}\right)\hat{\theta}^2_{r,\vec{p}} - \ldots + (-1)^r\left(\prod_{i_r=1}^{r} p_{i_r}\right)\hat{\theta}^r_{r,\vec{p}}. \tag{A.1}$$

This will be done by induction on the number of products. Let $r=1$, then

$$\prod_{i=1}^{1}(1-p_i\hat{\theta}_{1,\vec{p}}) = 1-p_1\hat{\theta}_{1,\vec{p}} = 1-\sum_{j=1}^{1} p_j\hat{\theta}_{1,\vec{p}}.$$

Suppose for some $k\in\mathbb{N}$, $k\ge 1$, if $r=k$, then

$$\prod_{i=1}^{k}(1-p_i\hat{\theta}_{k,\vec{p}}) = 1-\left(\sum_{i_1=1}^{k} p_{i_1}\right)\hat{\theta}_{k,\vec{p}} + \left(\sum_{1\le i_1<i_2\le k} p_{i_1}p_{i_2}\right)\hat{\theta}^2_{k,\vec{p}} - \ldots + (-1)^k\left(\prod_{i_k=1}^{k} p_{i_k}\right)\hat{\theta}^k_{k,\vec{p}}.$$

Now let $r = k+1$, so

$$
\begin{aligned}
\prod_{i=1}^{k+1}(1-p_i\hat{\theta}_{k+1,\vec{p}}) &= \left[\prod_{i=1}^{k}(1-p_i\hat{\theta}_{k+1,\vec{p}})\right](1-p_{k+1}\hat{\theta}_{k+1,\vec{p}}) \\
&= \left[1-\left(\sum_{i_1=1}^{k} p_{i_1}\right)\hat{\theta}_{k+1,\vec{p}} + \left(\sum_{1\le i_1<i_2\le k} p_{i_1}p_{i_2}\right)\hat{\theta}^2_{k+1,\vec{p}} - \ldots + (-1)^k\left(\prod_{i_k=1}^{k} p_{i_k}\right)\hat{\theta}^k_{k+1,\vec{p}}\right](1-p_{k+1}\hat{\theta}_{k+1,\vec{p}}) \\
&= 1-\left(\sum_{i_1=1}^{k} p_{i_1}\right)\hat{\theta}_{k+1,\vec{p}} + \left(\sum_{1\le i_1<i_2\le k} p_{i_1}p_{i_2}\right)\hat{\theta}^2_{k+1,\vec{p}} - \ldots + (-1)^k\left(\prod_{i_k=1}^{k} p_{i_k}\right)\hat{\theta}^k_{k+1,\vec{p}} \\
&\quad - p_{k+1}\hat{\theta}_{k+1,\vec{p}} + p_{k+1}\left(\sum_{i_1=1}^{k} p_{i_1}\right)\hat{\theta}^2_{k+1,\vec{p}} - p_{k+1}\left(\sum_{1\le i_1<i_2\le k} p_{i_1}p_{i_2}\right)\hat{\theta}^3_{k+1,\vec{p}} \\
&\quad + \ldots + (-1)^{k+1}p_{k+1}\left(\prod_{i_k=1}^{k} p_{i_k}\right)\hat{\theta}^{k+1}_{k+1,\vec{p}} \\
&= 1-\left(\sum_{i_1=1}^{k} p_{i_1}\right)\hat{\theta}_{k+1,\vec{p}} - p_{k+1}\hat{\theta}_{k+1,\vec{p}} + \left(\sum_{1\le i_1<i_2\le k} p_{i_1}p_{i_2}\right)\hat{\theta}^2_{k+1,\vec{p}} + p_{k+1}\left(\sum_{i_1=1}^{k} p_{i_1}\right)\hat{\theta}^2_{k+1,\vec{p}} \\
&\quad - \left(\sum_{1\le i_1<i_2<i_3\le k} p_{i_1}p_{i_2}p_{i_3}\right)\hat{\theta}^3_{k+1,\vec{p}} - p_{k+1}\left(\sum_{1\le i_1<i_2\le k} p_{i_1}p_{i_2}\right)\hat{\theta}^3_{k+1,\vec{p}} \\
&\quad + .. + (-1)^k\left(\prod_{i_k=1}^{k} p_{i_k}\right)\hat{\theta}^k_{k+1,\vec{p}} + (-1)^k p_{k+1}\left(\sum_{1\le i_1<i_2<\ldots<i_{k-1}\le k} p_{i_1}p_{i_2}\ldots p_{i_{k-1}}\right)\hat{\theta}^3_{k+1,\vec{p}} \\
&\quad + (-1)^{k+1}p_{k+1}\left(\prod_{i_k=1}^{k} p_{i_k}\right)\hat{\theta}^{k+1}_{k+1,\vec{p}} \\
&= 1-\left(\sum_{i_1=1}^{k} p_{i_1} + p_{k+1}\right)\hat{\theta}_{k+1,\vec{p}} + \left(\sum_{1\le i_1<i_2\le k} p_{i_1}p_{i_2} + p_{k+1}\sum_{i_1=1}^{k} p_{i_1}\right)\hat{\theta}^2_{k+1,\vec{p}} \\
&\quad - \left(\sum_{1\le i_1<i_2<i_3\le k} p_{i_1}p_{i_2}p_{i_3} + p_{k+1}\sum_{1\le i_1<i_2\le k} p_{i_1}p_{i_2}\right)\hat{\theta}^3_{k+1,\vec{p}} + \ldots + \\
&\quad (-1)^k\left(\prod_{i_k=1}^{k} p_{i_k} + p_{k+1}\sum_{1\le i_1<i_2<\ldots<i_{k-1}\le k} p_{i_1}p_{i_2}\ldots p_{i_{k-1}}\right)\hat{\theta}^3_{k+1,\vec{p}} \\
&\quad + (-1)^{k+1}p_{k+1}\left(\prod_{i_k=1}^{k} p_{i_k}\right)\hat{\theta}^{k+1}_{k+1,\vec{p}} \\
&= 1-\left(\sum_{i_1=1}^{k+1} p_{i_1}\right)\hat{\theta}_{k+1,\vec{p}} + \left(\sum_{1\le i_1<i_2\le k+1} p_{i_1}p_{i_2}\right)\hat{\theta}^2_{k+1,\vec{p}} - \left(\sum_{1\le i_1<i_2<i_3\le k+1} p_{i_1}p_{i_2}p_{i_3}\right)\hat{\theta}^3_{k+1,\vec{p}} \\
&\quad + .. + (-1)^k\left(\sum_{1\le i_1<i_2<\ldots<i_k\le k+1} p_{i_1}p_{i_2}\ldots.p_{i_k}\right)\hat{\theta}^3_{k+1,\vec{p}} + (-1)^{k+1}p_{k+1}\left(\prod_{i_k=1}^{k} p_{i_k}\right)\hat{\theta}^{k+1}_{k+1,\vec{p}}.
\end{aligned}
$$

Thus by the principle of mathematical induction (A.1) holds.

Appendix B

Statement of the Implicit Function Theorem

The following version of the Implicit Function Theorem is taken from [3] (page 107). This result is used in Section 2.5.

Let $G : \mathbb{R}^{r+1} \to \mathbb{R}$ be a continuously differentiable, and suppose that $G(\mathbf{a}) = 0$ while $\dfrac{\partial G(\mathbf{a})}{\partial x_{r+1}} \neq 0$. Then there exists a neighborhood U of $\mathbf{a}$ and a differentiable function F defined on a neighborhood V of $(a_1, \ldots, a_r) \in \mathbb{R}^r$, such that

$$U \cap G^{-1}(0) = \{\mathbf{x} \in \mathbb{R}^{r+1} : (x_1, \ldots, x_r) \in V \text{ and } x_{r+1} = F(x_1, \ldots, x_r)\}.$$

In particular,

$$G(x_1, \ldots, x_r, F(x_1, \ldots, x_r)) = 0$$

for all $(x_1, \ldots, x_r) \in V$.

Appendix C

Known critical values and conditions for critical surfaces of lattices

We summarize several results concerning critical values and critical surfaces for percolation model on lattices used in Chapter 3. These results are taken from [2].

The critical points of known lattices in the homogeneous case are as follows (see page 53 in [2]):

square lattice $p_c(\mathbb{L}^2) = \frac{1}{2}$

triangular lattice $p_c(\mathbb{T}) = 2\sin(\pi, 18)$

hexagonal lattice $p_c(\mathbb{H}) = 1 - 2\sin(\pi, 18)$

bow-tie lattice, $\mathbb{B}$, is the unique root in (0,1) of $1 - x - 6x^2 + 6x^3 - x^5 = 0$ ($p_c(\mathbb{B}) \approx 0.404518$)

Theorem 11.115: Critical surface of the inhomogeneous square lattice.

For the inhomogeneous square lattice and $\mathbf{p} = (p_h, p_v)$ with $p_h, p_v \in [0,1)$, the probability $\Theta_{\mathbb{L}^2}(\mathbf{p})$ that the origin percolates satisfies

$$\Theta_{\mathbb{L}^2}(\mathbf{p}) \begin{cases} = 0, & if\ \phi(\mathbf{p}) \leq 1, \\ > 0, & if\ \phi(\mathbf{p}) > 1, \end{cases}$$

where $\phi(\mathbf{p}) = p_v + p_h$. If p_h or $p_v = 1$, then $\Theta_{\mathbb{L}^2}(\mathbf{p}) = 1 > 0$. Therefore, the critical surface is

$$p_v + p_h = 1. \tag{C.1}$$

Remark 1: So, we observe that for some parameter values $\mathbf{p}$ lying on the critical surface $\Theta_{\mathbb{L}^2}(\mathbf{p}) > 0$ and for others $\Theta_{\mathbb{L}^2}(\mathbf{p}) = 0$.

Theorem 11.116: Critical surface of the inhomogeneous triangular lattice.

For the inhomogeneous triangular lattice and $\mathbf{p} = (p_h, p_v, p_d)$, with $p_h, p_v, p_d \in [0,1)$, the probability $\Theta_{\mathbb{T}}(\mathbf{p})$ that the origin percolates satisfies

$$\Theta_{\mathbb{T}}(\mathbf{p}) \begin{cases} = 0, & if\ \psi(\mathbf{p}) \leq 1, \\ > 0, & if\ \psi(\mathbf{p}) > 1, \end{cases}$$

where $\psi(\mathbf{p}) = p_v + p_h + p_d - p_v p_h p_d$. If p_h, p_v or $p_d = 1$, then $\Theta_{\mathbb{L}^2}(\mathbf{p}) = 1 > 0$. Therefore, the critical surface is the set of parameters that satisfies

$$p_v + p_h + p_d - p_v p_h p_d = 1. \tag{C.2}$$

A remark similar to Remark 1 holds for bond percolation on the triangular lattice.

Note that when any one of the three parameters is equal to 1, then there automatically is percolation. For instance, if $p_h = 1$, then every edge along every horizontal line is open. Similarly, for vertical lines when $p_v = 1$ and diagonal lines when $p_d = 1$.

Corollary to Theorem 11.116: Critical surface of the inhomogeneous hexagonal lattice.

For the inhomogeneous triangular lattice and $\mathbf{p} = (p_h, p_v, p_d) \in (0,1]^3$, where the edges in the triangular lattice are open for the horizontal, vertical, and diagonal edges with probabilities p_h, p_v and p_d respectively, and the dual lattice, i.e., the hexagonal lattice, which has corresponding open edge probabilities $1 - p_h$, $1 - p_v$, and $1 - p_d$ respectively, the probability $\Theta_{\mathbb{H}}(\mathbf{1} - \mathbf{p})$ that the origin of the hexagonal lattice percolates satisfies

$$\Theta_{\mathbb{H}}(\mathbf{1} - \mathbf{p}) \begin{cases} = 0, & if\ \chi(\mathbf{p}) \geq 1, \\ > 0, & if\ \chi(\mathbf{p}) < 1, \end{cases}$$

where $\chi(\mathbf{p}) = (1-p_v)+(1-p_h)+(1-p_d)-(1-p_v)(1-p_h)(1-p_d)$ and $\mathbf{1}$ is the vector with all entries equal to 1.

Therefore, the critical surface for the hexagonal lattice with parameters $(1-p_h, 1-p_v, 1-p_d)$ is the set of parameters that satisfy

$$(1-p_v)+(1-p_h)+(1-p_d)-(1-p_v)(1-p_h)(1-p_d) = 1, \tag{C.3}$$

where $p_h, p_v, p_d \in (0,1]$.

Note that when any one of these parameters is equal to 0, say $p_v = 0$, then the only edges, that have a possibility of being open are along strands of alternating horizontal and diagonal bonds. In this case, percolation can only occur if $p_h = p_d = 1$. Also, note that when any one of these parameters is equal to 1, say $p_d = 1$, then the resulting percolation model on the hexagonal lattice is equivalent to an inhomogeneous percolation model on the square lattice with parameters p_h and p_v, and the critical surface given in (C.3) reduces to the surface (C.1) described in Theorem 11.115: $p_v + p_h = 1$.

The result in the corollary is analogous to what holds for inhomogeneous bond percolation on the square lattice. Indeed, just as (p_v, p_h) is on the critical surface for the inhomogeneous percolation on the square lattice if and only if $(1-p_v, 1-p_h)$ is also on the critical surface - since $(1-p_v, 1-p_h)$ are the parameters for the dual model and the square lattice is self-dual - (p_h, p_v, p_d) is on the critical surface for the inhomogeneous hexagonal model if and only if $(1-p_h, 1-p_v, 1-p_d)$ is on the critical surface for the inhomogeneous triangular model.

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