CALIFORNIA STATE UNIVERSITY SAN MARCOS

THESIS SIGNATURE PAGE

THESIS SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE

MASTER OF SCIENCE

IN

MATHEMATICS

THESIS TITLE: English Translation of the *Sphaerica* of Menelaus

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DATE OF SUCCESSFUL DEFENSE: December 2, 2015

THE THESIS HAS BEEN ACCEPTED BY THE THESIS COMMITTEE IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE IN MATHEMATICS.

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English Translation of the Sphaerica of Menelaus

Rani Hermiz

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Special Thanks to James Brust.

Abstract

The **Sphaerica** (in English: Spherics) of Menelaus of Alexandria (dating to roughly 100 AD) is among the oldest known works on spherical geometry and trigonometry. Spherical geometry is the study of geometric objects on the surface of a sphere and spherical trigonometry is the study of relationships among sides and angles in triangles on a sphere, where the sides are arcs of "great circles." The **Sphaerica** was originally written in Koiné Greek, but editions in this language are no longer extant. One of the oldest complete editions still available is Abu Nasr Mansur's "improved" edition in Arabic. Other editions exist in Arabic, Hebrew, Latin, and German, but none in English. In this thesis I give an English translation of Abu Nasr Mansur's edition, and discuss certain aspects of the text from a modern point of view.

1. Historical Background

Spherical geometry and trigonometry were originally developed in the context of astronomy. Ancient peoples observing the heavens were able to notice that the stars seemed to move in parallel circular orbits, centered on a fixed point in the sky. Without a way of knowing that the Earth is constantly rotating, which would have accounted for the apparent motion of the stars and partly accounted for the motion of the Sun, Moon, and planets, people naturally developed the idea that these bodies were all attached to a *celestial sphere*, centered at the Earth, that rotated around an axis passing through the Earth.

Although the Sun also seemed to rotate with the celestial sphere, over a period of several days a careful observer would have noticed that the sphere seemed to be in a slightly different position each new time the sun set. Between this and being aware of the changing seasons, astronomers came to view the Sun as traveling around a circle on the celestial sphere that we today call the *ecliptic*. The fact that the Earth's orbit around the Sun stays (essentially) in the same plane causes the Sun's apparent motion around the Earth to stay in a plane, and therefore on a circle: the intersection of a sphere with a plane that is not tangent to it is always a circle. The intersection of the sphere with a plane containing the center is called a *great circle* because such circles are the largest ones on the sphere. Since the Earth's center is revolving around the Sun's center, the plane of the Sun's apparent motion goes through the center of the celestial sphere, so the ecliptic is a great circle of the celestial sphere.

Great circles are part of the foundation of spherical geometry. An important fact in spherical geometry is that a minimum-length path between any two points on a sphere is an arc of a great circle crossing through these two points. If these points are not *antipodal*, i.e. not on opposite ends of a diameter of the sphere, then there is only one great circle through them and therefore a unique minimum-length path between them; otherwise there are infinitely many great circles passing through them. Thus great circles are the *geodesics*—the analogues of straight lines—in spherical geometry. An important distinction between great circles on the sphere and lines in the plane is that a pair of great circles can never be parallel: they always intersect in exactly one pair of antipodal points. Two of the most important points on the celestial sphere are the points where the ecliptic intersects a certain other great circle.

The *celestial equator* is the great circle whose plane is perpendicular to the axis of the Earth's rotation (one can also think of it as the great circle directly above the Earth's equator, which is a great circle on the Earth's surface). The points at which the ecliptic crosses the celestial equator are called the *equinoxes* because day and night are of roughly equal length when the Sun is crossing one of the equinoxes.[1] For an observer not on the Earth's equator, the period of daylight would be longest or shortest as the Sun passes through one of the two *solstices*, the points on the ecliptic most distant from the celestial equator.

Because the Earth and the other planets are in reality revolving around the Sun on roughly the same plane, each planet keeps within a few degrees of the ecliptic in its apparent motion through the celestial sphere. Thus early astronomical cultures such as the Babylonians and very early Greek astronomers tended to focus on the *Zodiac*, a band around the ecliptic divided into twelve 30° arcs. However, these early cultures tended to use very little geometry in their astronomical reckoning, instead relying on arithmetic models that they had put together from observation.[2]

By the mid-fourth century BC, Greeks began to incorporate more spherical geometry into their astronomical works. Books by Autolycus of Pitane in the late fourth century, and Euclid about a generation later,

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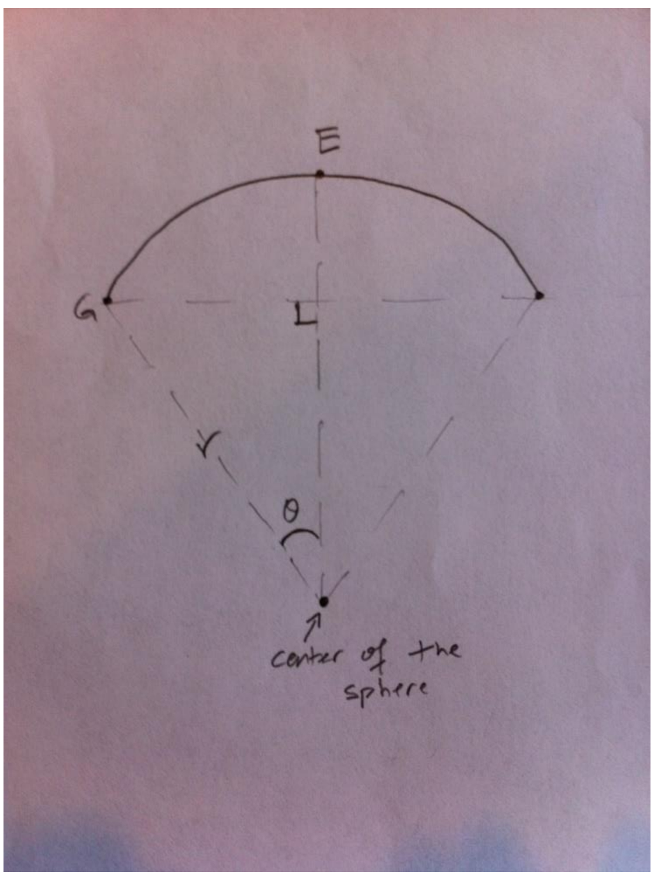
[1] Since the Sun's image in the sky is larger than just a point, the day is in fact slightly longer than night when the Sun's center crosses the equinoxes. Atmospheric refraction is also a factor. Therefore, the closest thing to a true "equinox" would occur slightly before the vernal (Spring) equinox and slightly after the autumnal equinox.

[2] The principal reference for the astronomical and trigonometric background is [VB1].

used purely geometrical reasoning to solve astronomical problems. In the late second century BC, Theodosius of Bithynia compiled the existing Greek knowledge of spherical geometry in his **Sphaerica**, with many propositions that were essentially abstract versions of astronomical problems. Among the Greek works that survive, Theodosius' **Sphaerica** seems to best represent the state of Greek spherical geometry before the development of spherical trigonometry, although van Brummelen [VB1] mentions evidence suggesting that Hipparchus of Nicaea used some trigonometry himself in the mid- to late second century BC. Hipparchus' writings are mostly gone, but Claudius Ptolemy's **Almagest** (c. 150 AD) refers extensively to his works.

At this point we should be very clear about what kind of "trigonometry" existed in ancient Greek mathematics. The trigonometric "functions" as we think of them today did not exist. However, as early as Hipparchus there were mathematicians compiling tables of approximations for the lengths of chords corresponding to certain arc-lengths, and applying these to astronomical problems. Given a circle of radius 1 and an arc of that circle of length θ, let $Crd(\theta)$ denote the length of the line segment between that arc's endpoints. This function may be regarded as a primitive trigonometric function.



When the given circle has radius R, an arc of length θ subtends an angle of measure $\frac{\theta}{R}$. So in the case that $R = 1$ we can see that

$$\sin(\theta) = \frac{1}{2} Crd(2\theta).$$

In fact, the earliest form of the sine function was invented by Indian mathematicians to replace the chord function, and thus eliminate the unnecessary steps of doubling arcs and halving their chords which often came up in Greek uses of the chord function.

One of the astronomy solutions in Ptolemy's **ALMAGEST** uses chord approximations to determine the *eccentricity* of the Sun's apparent orbit, the distance from the Earth to the center of the regular circular orbit which would match the Sun's motion. Menelaus' **SPHAERICA** contains many demonstrations by which one could apply chord tables—or sine tables—to the astronomical problems of his day.

Menelaus of Alexandria lived from about 70 AD to 140 AD. Although only fragments of his other writings survive, we know that he wrote geometrical treatises called **ELEMENTS OF GEOMETRY** and **ON THE TRIANGLE**, and at least part of a star catalogue. Something of his mathematical personality can be seen in the fact that he attempted to give a direct proof in almost every demonstration of the **SPHAERICA,** avoiding even the most obvious contradiction proofs.[3]

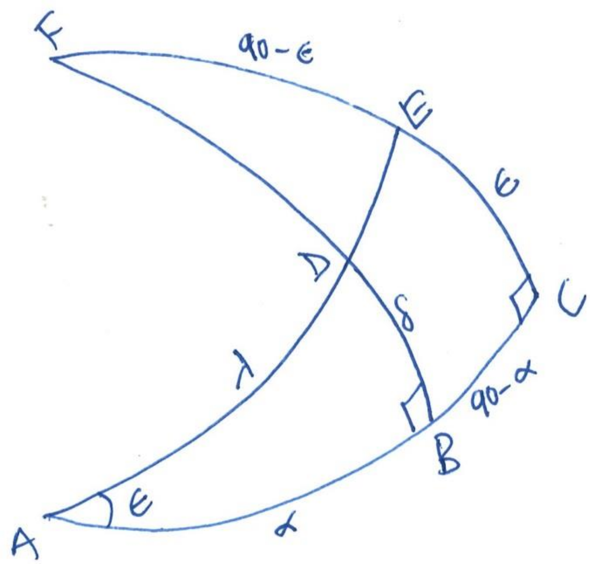
Menelaus wrote the original text in Greek, but this text has been lost. There were three Arabic translations, all of which have been lost. These date to around the eighth or ninth centuries. The text for this translation is known as the Abu Nasr Mansur version of the text, and is the result of reworking of earlier translations. Various versions of **SPHAERICA** have also been translated into Hebrew (by Jacob ben Maher in the thirteenth century) and Latin (by Gerhard of Cremona in the twelfth century and Edmund Halley in 1758). The Abu Nasr Mansur version was translated into German by Krause[K] in 1936. Until recently, no English version of **SPHAERICA** has appeared. (See [S],[K],[L].)

Arabic editions differ slightly as to the numbering of the demonstrations in the **SPHAERICA** and as to where Book I ends and Book II begins, but every known edition divides the **SPHAERICA** into three books, and the breakdown is roughly as follows:

[3] Examples of this are in the next section.

- Book I establishes the basic geometry of triangles on the sphere. Several of the results proven here are analogous to well-known facts about planar triangles, and there are many others that are unique to spherical triangles.
- Book II begins to prove geometric facts directly relating to astronomy. Menelaus himself never seems to mention the astronomical context of his demonstrations—he states them as purely geometrical results—but in some cases they seem to be of little use or interest outside their astronomical applications, which Abu Nasr Mansur lays out in his commentaries.
- Book III begins to use trigonometry and apply it to more advanced astronomical problems.

The first demonstration of Book III, which some attribute to Menelaus and now often referred to as `Menelaus' theorem', was used by Ptolemy in solving all of the important spherical astronomy problems in the **ALMAGEST**, and its corollaries that follow prove to be foundational in the spherical astronomy of the medieval Islamic world.



Referring to the above diagram, the modern version of Menelaus' theorem asserts that

$$\frac{\sin(\widehat{AB})}{\sin(\widehat{BC})} \cdot \frac{\sin(\widehat{CF})}{\sin(\widehat{FE})} \cdot \frac{\sin(\widehat{ED})}{\sin(\widehat{DA})} = 1.$$

Menelaus' theorem can be used in charting the Sun's position. It is especially useful for translating between different coordinate systems. In the *equatorial coordinate system*, an object's position on the celestial sphere is given in terms of its *declination* δ, which is its angular distance from the equator, and its *right ascension* α, which is the right-handed angle corresponding to the shorter arc of the equator between the vernal equinox and the object.[4] Let us see how to use Menelaus' theorem to chart the sun's position. We have,

$$\frac{\sin \alpha}{\sin(90 - \alpha)} \cdot \frac{\sin 90}{\sin(90 - \epsilon)} \cdot \frac{\sin(90 - \lambda)}{\sin \lambda} = 1,$$

So $\alpha = \tan^{-1}(\tan \lambda \cdot \cos \boldsymbol{\epsilon})$. We also have

[4] In modern usage, of course, these coordinates can be positive or negative. Menelaus, Abu Nasr, and contemporaries of them dealt with these concepts as the lengths of arcs and always took them to be positive, but the theorem is still correct in the modern usage.

$$\frac{\sin(90-\epsilon)}{\sin\epsilon}\cdot\frac{\sin 90}{\sin\alpha}\cdot\frac{\sin\delta}{\sin(90-\delta)}=1,$$

and so we get that the sun's position $\delta = \tan^{-1}(\sin\alpha \cdot \tan\epsilon)$.

There is some dispute as to whether Menelaus' theorem (called the *sector theorem* or *transversal theorem* in medieval Islamic books) was really first proven by Menelaus. Sidoli [S] interprets the role of this theorem in the **Sphaerica** as suggesting that Menelaus never intended anyone to think of it as his own, and Sidoli also mentions evidence suggesting that Hipparchus knew of and used this theorem almost two and a half centuries earlier. (It is hard to say for sure, because so little of Hipparchus' writing has survived. We know of it mainly from the references that other writers have made to it.)

A lot happened in the development of trigonometry and spherical astronomy in the nine centuries between the original **Sphaerica** and the Arabic edition translated here. Ptolemy apparently reached the apex of Greek work in those areas only a generation or two after Menelaus. Roughly two hundred years later (in the fourth or fifth century AD) Indian astronomers were writing texts that focused on calculation (usually presented in mnemonic verse) while presenting little or no geometry, but contained methods that the Greeks had used geometry to derive. The earliest Indian works that are still extant, from about the sixth century, contain sine tables and totally lack chord tables. We also know that the cosine function was used as early as the fifth century. Indian mathematicians developed several novel ways of calculating sines, including a method for calculating successive differences between the sines of consecutive multiples of 3° 45', and Bhāskara I's rational approximation[5]

$$\sin(\theta) \cong \frac{4\theta(180^\circ-\theta)}{40500^\circ-\theta(180^\circ-\theta)},$$

and many trigonometric identities, such as the formula for the sine of the sum or difference of two given arcs.

Indian astronomy passed into the Islamic world as early as the eighth century, and it was there that the true successors to the Greeks would emerge, as far as applying spherical geometry to astronomy was concerned. At some point in the ninth century, Islamists were translating Greek treatises on mathematics and science—as well as others in Sanskrit and Syriac—into Arabic. New astronomical works

[5] [VB1], p. 103.

were generally inspired by the works of Ptolemy. In addition to the usual concerns of early astronomers, Islamist spherical geometers were interested in ways of determining such things as the *qibla*—the direction pointing from a person's location toward the Kaaba in Mecca, in which direction a Muslim was expected to pray—and the prescribed periods of time for each of the five daily prayers.

Abū Naṣr Manṣūr ibn ʿAlī ibn ʿIrāq was born in Gilan (now in Iran) around 950 AD and died in 1036 in Ghazna (now in Afghanistan). Contemporaries called him "the prince", which suggests that he was one of the Banū ʿIrāq dynasty that ruled Khwarezm until they were violently overthrown in 995. According to **The Biographical Encyclopedia of Astronomers**[H], Abu Nasr claimed credit for discovering the spherical and planar laws of sines; he used the spherical law of sines without proof many times in his edition of the **Sphaerica**[6], and apparently gave several proofs in his other writings. He also is said to have discovered the concept of a "polar triangle" for each spherical triangle, a triangle whose vertices are poles of the given triangle's sides.[7] A basic fact about the correspondence between a triangle's sides and its polar triangle's interior angles—and vice versa—is a powerful tool in "solving" triangles, i.e. using given measurements of sides and/or angles in a triangle to find others, because sometimes known methods will apply much more easily to the polar triangle than the original triangle. This correspondence also lets one prove a few of the early demonstrations in Book I of the **Sphaerica** as trivial corollaries of earlier ones.

Abu Nasr is said to have also written his own astronomical masterwork, **The Royal Almagest**, and a work on methods of finding the qibla, *The* **Book of Azimuths**, but these are both almost entirely lost. His surviving writings fall into two categories, those concerning astrolabes—which in those days were used for navigation, timekeeping, and finding the qibla—and commentaries on, or reactions to, the works of others. He seems to have had fairly conservative views in astronomy: for instance, he is said to have firmly supported the mainstream idea that all heavenly bodies traveled with uniform, circular orbits, criticizing a

[6] This implies that the sine function was already applied to angles as well as arcs—so widely that Abu Nasr could expect his reader to understand this usage.

[7] The full definition also requires that, of the two poles of each side in the triangle, we choose the pole that is on the same side of that arc as the vertex of the angle that this arc is subtending in the original triangle.

colleague's suggestion that perhaps the planets' orbits were ellipses that very nearly approximated circles.[8]

This translation is mostly literal in the sense that it does not attempt to insert modern language or formatting. In particular, the word 'demonstration' can be thought of as a combined theorem/proof format. In most 'demonstrations', the author states a proposition without notation, restates it with notation, and then proves the proposition. There are some deviations from this format. For example, Demonstration 15 can be seen as the proof for a second case of the propositions stated in Demonstration 14. Similarly, Demonstrations 20 and 21 in book I are proofs of cases of the proposition stated in Demonstration 19.

[8] Much of the material from the last two paragraphs is from [H].

2. A Modern Perspective on Menelaus' Mathematics

There are several noticeable differences between the text's approach toward results in spherical geometry and approaches that would be typical in a modern setting. Some of these differences are due to the fact that modern mathematicians have access to ideas that the author did not, but others seem like they can be attributed to the author's own idiosyncrasies.

Consider a spherical triangle ABC. As one might do in planar trigonometry, let us use A, B, and C to denote the measures of the angles with vertices A, B, and C, respectively; and let us use the lower-case letters a, b, and c to denote the measures of the sides opposite the vertices A, B, and C, respectively. The text of **SPHAERICA** generally starts with the letters A,B,G,D,E,Z to label triangles, but generally we will use the letters A,B,C,D,E,F, as is done in English. In particular we restate the demonstrations using A,B,C,D,E,F.

Proofs by contradiction (known to ancient Greeks as "reduction to the impossible") are far from being strictly modern: for example, there are contradiction proofs in Euclid's **ELEMENTS**. The avoidance of contradiction proofs seems to be a matter of personal preference. An obvious example is his proof of Book I, Demonstration 9: *Between any two sides of a triangle, a greater side subtends a greater angle.* (For example, if a > b then A > B.) Consider the following demonstrations that have already been proven:

Demonstration 2: If a = b then A = B.

Demonstration 3: If A = B then a = b.

Demonstration 7: If A > B then a > b.

One could then prove Demonstration 9 as follows:

Assume, for the sake of contradiction, that a > b but A is not greater than B. Then A is less than or equal to B. If A < B then by Demonstration 7 it follows that a < b, which is a contradiction. If A = B then by Demonstration 3 we have a = b, which is a contradiction. Therefore A must be greater than B. □

Menelaus' proof is based on a fairly simple construction and is only a little longer than our proof above, but our proof illustrates the more basic logical connections between Demonstrations 2, 3, 7, and 9.

Another example of a result that could have been more easily proven by contradiction is Demonstration 25: *If one angle in a triangle is not less than a right angle and each of the two sides enclosing one of the two remaining angles is less than a quarter-circle, then the remaining side is less than a quarter-circle and each of the remaining angles is acute.* In the case of triangle ABC we restate that as: "Suppose $C \geq 90°$, $b < 90°$, and $c < 90°$. Then $a < 90°$, $A < 90°$, and $B < 90°$."

In Demonstration 24 Menelaus has already proven what we can write as: "Suppose $C \geq 90°$, $a < 90°$, and $b < 90°$. Then $A < 90°$ and $B < 90°$." This lends itself to a straightforward contradiction proof for Demonstration 25:

Suppose $C \geq 90°$, $b < 90°$, and $c < 90°$; and assume for the sake of contradiction that $a \geq 90°$. Then $a > c$, and so $A > C$ by Demonstration 9. Therefore $A > 90°$. Since it is given that $b < 90°$ and $c < 90°$, by Demonstration 24 it follows that $C < 90°$, which is a contradiction. Thus, $a < 90°$. It follows from Demonstration 24 that $A < 90°$ and $B < 90°$. □

One of the helpful notions used for some purposes in the text is the notion of co-lunar triangle. Given a point X on the sphere, we let X^a denote the antipode of X. Then triangle ABC has three co-lunar triangles: triangle A^aBC, triangle AB^aC and triangle ABC^a.

Consider the triangle A^aBC, which is colunar with triangle ABC. The side of A^aBC that subtends angle A^a is the same as the one that subtends the angle A in triangle ABC, so has the same measure. The side of A^aBC that subtends angle B forms a great semicircle with the side of ABC that subtends angle B in that triangle, so the side opposite B in triangle A^aBC has measure $180° – b$ in triangle A^aBC. The angle BA^aC in triangle A^aBC has the same measure as angle A in triangle ABC, because these are formed by the same two great semicircles. The angle A^aBC forms a linear pair with the angle ABC, and therefore they are supplementary, so the measure of angle A^aBC is $180° – B$.

The text uses the notion of co-lunar triangle to provide a straightforward way of proving Demonstration 10 from earlier demonstrations. Let us restate Demonstrations 2, 3, 7, and 9 as follows:

Demonstration 2: If $a – b = 0$ then $A – B = 0$

Demonstration 3: If $A - B = 0$ then $a - b = 0$

Demonstration 7: If $A - B > 0$ then $a - b > 0$.

Demonstration 9: If $a - b > 0$ then $A - B > 0$.

Using the modern notion of positive and negative numbers, we can put these demonstrations together and conclude that $a - b$ has the same sign as $A - B$ (or both are zero), and use in the co-lunar triangles. (For simplicity, let us call this the “subtending sides theorem”.) Now we have an easy proof of Demonstration 10, which in rather long terms states that $a + b - 180°$ has the same sign as $A + B - 180°$(or both are zero). The proof follows:

Consider the triangle A^aBC, which is colunar with triangle ABC. Applying the subtending sides theorem to triangle A^aBC, we see that $a - (180° - b)$ has the same sign as $A - (180° - B)$. In other words, $a + b - 180°$ has the same sign as $A + B - 180°$. □

However, the text does not use colunar triangles in another interesting case. If we instead use the triangle inequality on triangle A^aBC, we find that $(180°-b)+(180°-c)>a$, so $a+b+c<360°$. This result does not appear in the text.

The notion of polar triangle is something that (as far as we know) had not yet been invented in the time of Menelaus. Although Abu Nasr is credited with inventing them, they were never used in his translation. Let ABC be a spherical triangle. We define the polar triangle $A'B'C'$ of triangle ABC as follows. Let A' be the pole which lies on the same side of great circle BC as A. We define B' and C' analogously: B' is the pole of great circle AC on the same side of great circle AC as B, and C' is the pole of great circle AB on the same side of great circle AB as C. Then A', B' and C' form a well-defined spherical triangle whose angles and sides have measures denoted by A',B',C',a',b' and c'. We have the following important theorem for the polar triangle.

Theorem∗: If ABC is a spherical triangle and $A'B'C'$ is its polar triangle, then

$$180° - \mathrm{a} = \mathrm{A}'$$

$$180° - \mathrm{b} = \mathrm{B}'$$

$$180° - \mathrm{c} = \mathrm{C}'$$

$$180° - \mathrm{C} = \mathrm{c}'$$

$$180° - \mathrm{B} = \mathrm{b}'$$

$$180° - \mathrm{A} = \mathrm{a}'$$

One thing this powerful theorem immediately allows us to do is derive certain congruence theorems from others. For example, Demonstration 18 proves the surprising result that if two triangles have all three pairs of corresponding angles congruent, then the triangles are congruent[9] (the 'AAA congruence theorem'.) Demonstration 4 proves what amounts to the SSS and SAS congruence theorems for spherical triangles[10], so one who knows about polar triangles can prove Demonstration 18 as follows:

Suppose triangle ABC and DEF are such that A = D, B = E, and C = F. Then in the polar triangles A′B′C′ and D′E′F′ we have a′ = d′, b′ = e′, and c′ = f′, by the above theorem. By Demonstration 4, triangle A′B′C′ is congruent to triangle D′E′F′, so A′ = D′, B′ = E′, and C′ = F′. It follows from Theorem ∗ that a = d, b = e, and c = f; so by Demonstration 4, triangle ABC and triangle DEF are congruent. □

Demonstration 14 states that an ASA congruence correspondence between two triangles is enough to conclude that the triangles are congruent. One may use similar method involving polar triangles to prove this from the SAS congruence theorem. Again, the text uses another more complicated method of proof which continues into demonstration 15.

Demonstration 13 proves that given two triangles ABC and DEF such that A = D, b = e, a = d and B + E ≠ 180˚ then triangles ABG and DEZ are congruent. Demonstration 17 assumes that A = D, C = F, a = d, and c + f ≠ 180˚ and proves that the triangles are congruent. So through the use of polar triangles, the proof of demonstration 17 would be as follows:

Suppose triangles ABC and DEF are such that A = D, C = F, a = d, and c + f ≠ 180˚. Then in the polar triangles A′B′C′ and D′E′F′ we have a′ = d′, c′ = f′, A′ = D′, and C′+ F′ ≠ 180˚. By demonstration 13, triangle A′B′C′ is

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[9] The text doesn't define the congruence of triangles, but does discuss the idea of when all corresponding sides and angles of a triangle are congruent.
[10] Technically, Menelaus only proves that SSS correspondence implies SAS correspondence, and vice versa; but this can obviously be applied to each pair of corresponding angles to show that either SSS correspondence or SAS correspondence would imply congruence.

congruent to triangle D′E′F′. From that we may conclude as before that triangle ABC and triangle DEF are congruent. □

We give one more example where polar triangles would provide a more modern proof. Demonstration 8 states that for the given two triangles ABC and DEF, if a = d, b = e and C < F then c < f. (This is frequently referred to as the Hinge theorem.) It is equivalent to write that if a = d, b = e, and C > F then c > f. Demonstration 4, in part, proves that c = f when a = d, b = e and C = F.

Let us see how we can use this to prove a portion of Demonstration 19. The first part of it asserts that if A = D, B = E and C > F then c > f. To prove this let's proceed as follows. We pass to the polar triangles A′B′C′ and D′E′F′ and apply Theorem ∗ to find that a′ = d′, b′ = e′ and 180˚– C < 180˚– F, or c′ < f′. We claim that C′ < F′. Proceeding by contradiction: if C′ = F′ then c′ =f′ by demonstration 4 (a contradiction.) If C′ > F′ then c′ > f′ by demonstration 8 (a contradiction.) Thus the only possibility is C′ < F′. By Theorem∗, this means that 180˚– c < 180˚– f and so c > f which is what we wanted to prove.

In Book III, the text uses the sine function, but not any other trigonometric function, presumably because the author of the text did not know those functions. Demonstration 3 illustrates the weakness of not having more trigonometric functions. Using letters, Demonstration 3 states the following: suppose that ABC and DEF are triangles with right angles at A and D, and the angles at C and F are equal but not right. Then

$$\frac{\sin c}{\sin \mathrm{b}} = \frac{\sin(\mathrm{f})}{\sin(\mathrm{e})} \cdot \frac{\sin(90-\mathrm{c})}{\sin(90-\mathrm{f})}.$$

The significance of this conclusion is not immediately clear. But if we allow ourselves the use of the cosine and tangent functions, the expression becomes simpler.

The factors sin (90 – f) and sin (90 – c) are sines of arcs complementary to DE and AB and we may write these as cos (f) and cos (c) in recent notations. So the expression becomes

$$\frac{\sin c}{\sin \mathrm{b}} = \frac{\sin(\mathrm{f})}{\sin(\mathrm{e})} \cdot \frac{\cos(\mathrm{c})}{\cos(f)}$$

Rearranging factors and introducing the tangent function,

$$\frac{\tan c}{\sin \mathrm{b}} = \frac{\tan(\mathrm{f})}{\sin(\mathrm{e})}.$$

But it is now a well-known part of the theory of spherical right triangles that in a spherical right triangle, the tangent of a (non-right) angle is found by taking the tangent of the opposite side divided by the sine of the adjacent side, so

$$\tan(C) = \frac{\tan(\mathrm{c})}{\sin(\mathrm{b})}$$

and

$$\tan(F) = \frac{\tan(\mathrm{f})}{\sin(\mathrm{e})}.$$

Thus in a roundabout way, Demonstration 3 shows how to calculate the tangent of an angle in terms of the sides of the spherical triangle.

3. Book I of Sphaerica

The figure that I call “triangle” is a figure in the surface of the sphere which is enclosed by three arcs of great circles, each of them less than a semicircle.

Its angles are the ones enclosed by these arcs. So one obtains the triangular surface, and the mentioned arcs enclose it.

The angles that I name equal angles are enclosed by great circles such that the arcs of inclination of their semicircles are equal, i.e. the arc between two circles of the circle that goes through both of their poles.

It is said that the angles evaluate those inclinations because they are poles of circles of those inclined ones.

It is said that great circles are perpendicular to each other if each of them goes through poles of the other.*The First Demonstration:* We would like to show how we can construct over a given arc of a great circle, from a certain point on it, an angle that is equal to a given angle enclosed by two great circles.

Let the given arc from a great circle be the arc AB, the given point be point B, and the given angle be the angle GDE. We would like to construct on the point B, from the arc AB, an angle equal to the angle GDE. We would like to construct on the point B, from the arc AB, an angle equal to the angle GDE.

With the point D as a pole we draw the arc GE. With B as a pole we draw an arc from A at the same distance from B as GE is from D, and we cut out from this arc the arc AZ, making it equal to GE. Since GE is equal to AZ then they subtend equal angles at the centers of their respective circles, and these are the arcs of inclination of the pairs of semicircles that we have mentioned, so angle ABZ is equal to angle GDE.

If the two circles of AZ and GE are great then they are equal. In the circle that goes through the poles of AB, BZ and the circle that goes through the poles of GD, DE, the [aforementioned] angles that are enclosed by diameters are the ones that end at A, Z, and G, E and they are equal; and these evaluate the angles at B and D.

If AZ and GE are not from the two great circles then they are parallel to the two great circles with B and D as poles; so the diameters that end at A and Z and the ones that end at G and E are parallel to diameters of the great circles that end in circles AB, BZ, and GD, DE separates parallel circles on their diameters.

The Second Demonstration: If a spherical triangle has two equal legs then the two angles on its base are equal.

Let triangle ABG be given with AB equal to BG. Then I say that angle BAG is equal to angle BGA. With the point A as a pole, we draw the arc GD. Similarly, we draw the arc AE with G as a pole. Then the arc AD is equal to the arc GE, so the arc BE is equal to the arc BD. So the arcs GD and AE, which are from equal circles, have equal arcs perpendicular to them at D and E, from which the equal arcs BE and BD have been cut off, and since BA and BG are equal then GD is equal to AE. Therefore, these two arcs subtend equal angles at the centers of their circles. GD is an arc of inclination for the angle BAG and the arc AE is an arc of inclination of the angle BGA, so angle BAG is equal to angle BGA.

The Third Demonstration: If two of the angles in a triangle are equal then the two sides that subtend them are equal.

Let the triangle ABG be given and let the angles at A and G be equal. Then I say that the side AB is equal to the side BG. With A and G as poles, respectively, we draw two great circles ZED and THD. Then D is a pole of the great circle TAGZ. The arc DHT is equal to the arc DEZ, and since the angle at G is equal to the angle at A then the arc TH is equal to the arc ZE, so DH is equal to DE. Then arcs BH and BE are segments of two equal circles that are perpendicular on the circles THD and ZED, respectively, so these segments are equal. So BA is equal to BG.

The Fourth Demonstration: Given two sides of a triangle equal to two sides of another triangle, if their bases are equal then the angles subtended by the bases are equal. Also, if the angles subtended by the bases are equal then the bases are equal.

Given the two triangles ABG and DEZ, let side AB be equal to side DE and side BG be equal to side EZ. I say that if base AG is equal to base DZ then the angle at B is equal to the angle at E, and if the angle at B is equal to the angle at E then base AG is equal to base DZ.

Using the points B and E as poles, we draw two arcs AH and DT, respectively. Because arc BG is equal to arc EZ and arc BH equal to arc ET then arc GH is equal to arc ZT. These equal segments are perpendicular to the equal circles AH and DT, and GA is equal to ZD, so the arc AH is equal to the arc DT. So, the angle at B is equal to the angle at E. And also, if the angle at B is equal to the angle at E then the arc AH is equal to the arc DT, and therefore AG is equal to DZ.

The Fifth Demonstration: In every triangle, the sum of any two sides is greater than the remaining side.

Let the triangle ABG be given. Then I say that the sum of any two sides is greater than the remaining side, whichever we choose.

Let BG be the greatest side. With the point B as a pole we make the circle AHD, and we finish drawing the circle BGED such that E is opposite to B. Then E is the other pole of AHD. Since GED is a great circle that goes through both poles of circle AHD then it is perpendicular to AHD and cuts it into two halves. The arc ED is equal to the arc EH, so the point G splits the arc DEH into two nonequal parts, with GH being shorter than GED. Since the circle HGED is perpendicular to the circle AHD then GH is the shortest arc that comes out from the point G to the arc AHD. So arc AG is greater than arc GH. Since arc AB is equal to arc BH then the sum of AB and AG is greater than BG.

If he didn't prefer the direct way then he could have proven it by contradiction. Given the angle ABG, let BG be the greatest side. Then I say that the sum of AB and AG is greater than BG. Assume for the sake of contradiction that BG is equal to the sum of AB and AG. Then we extend the arc AG through A to get arc AD equal to arc AG. We connect D and G with an arc from a great circle. Since AG is equal to AD then angle ADG is equal to the angle AGD, and since AG is equal to AD and BG equals the sum of AB and AG then BG is equal to the sum of AB and AD, which is equal to BD.

So angle BGD is equal to angle BDG, so the bigger angle BGD and the smaller angle AGD are equal, which is a contradiction. So BG is not equal to the sum of AB and AG, and this is what we wanted to show.

The Sixth Demonstration: If we construct on one side of a triangle two other sides that meet inside the triangle then their sum is less than the sum of the two other sides of the original triangle.

Given triangle ABG, draw the arcs AD and GD where D is a point inside the triangle. Then I say that the sum of AB and BG is greater than the sum of AD and DG.

If we extend GD to point H on the side AB then the sum of GB and BH is greater than GH by the fifth demonstration, and by adding AH to each of them we see that the sum of GB and BA is greater than the sum of GH and HA. Also by the fifth demonstration, the sum of AH and HD is greater than AD, so the sum of GH and HA is greater than the sum of GD and DA. So the sum of AB and BG is much greater than the sum of AD and DG.

The Seventh Demonstration: Between any two angles of a triangle, a greater angle is subtended by a greater side.

Let the triangle ABG be given and let the angle at point A be greater than the angle at point G. Then I say that the side BG is greater than the side AB.

Choose D on the side BG such that the angle DAG is equal to the angle BGA. Then AD is equal to DG, so by adding BD to both of them we see that BG is equal to the sum of AD and DB, which is greater than AB, so BG is greater than AB.

The Eighth Demonstration: Given two triangles such that two sides of one are equal to two sides of the other; and if the angle enclosed by these two sides in the first triangle is greater than the corresponding angle in the second triangle then the base of the first triangle is greater than the base of the second.

Let the two triangles ABG and DEZ be given and let AB be equal to DE and BG be equal to EZ. Then I say that if the angle at E is greater than the angle at B then the base DZ is greater than the base AG.

Choose H such that angle ABH is equal to the angle at E and BH is equal to EZ, and connect A and G to H. Then AH is equal to the base DZ. So in the first picture, since BH and BG are equal then angle BGH is equal to angle BHG, so angle AHG is smaller than angle BGH. Angle AGH is greater than angle BGH, so angle AGH is much greater than angle AHG, so the base AH is greater than the base AG. In the second picture we extend BG from G to J and extend BH from H to N. Since the two angles BGH and BHG are equal then the two angles JGH and GHN are equal. Since angle AHG is smaller than angle GHN and angle AGH is greater than angle JGH then angle AGH is much greater than AHG, so base AH is greater than base AG. The reverse can be shown by contradiction.

The Ninth Demonstration: Between any two sides of a triangle, a greater side subtends a greater angle.

Let the triangle ABG be given and let the side BG be greater than the side AB. Then I say that the angle at A is greater than the angle at G.

We choose D on BG such that DG is equal to AB and we draw a great-circle arc between A and D. Since the sum of AB and BD is greater than AD and AB is equal to DG then BG is greater than AD. Since AB is equal to DG and the triangles ABG and DGA have the side AG in common, and the side BG is greater than the side AD, then by the eighth demonstration the angle BAG is greater than the angle AGD, which is the angle AGB.

The Tenth Demonstration: If the sum of two sides of a triangle is less than a semicircle then the exterior angle behind one side is greater than the interior angle subtended by this side. If the sum of two sides of a triangle is greater than a semicircle then the exterior angle behind one side is less than the interior angle subtended by this side. If the sum of two sides of a triangle is equal to a semicircle then the exterior angle behind one side is equal to the interior angle subtended by this side.

Let the triangle ABG be given and let the sum of AB and BG be less than a semicircle. Then I say that the exterior angle BGD is greater than the angle BAG.

We extend the arcs AB and AG until they meet at D. Since the sum of AB and BG is less than BD, so by the ninth demonstration, the angle BDG, which equals the angle BAG, is less than the angle BGD.

Now let the sum of AB and BG be greater than a semicircle. Then I say that the exterior angle BGD is less than the angle BAG.

Since the sum of AB and BG is greater than BD, so by the ninth demonstration, the angle BDG, which equals the angle BAG, is greater than the angle BGD.

Now let the sum of AB and BG equal to a semicircle. Then I say that the exterior angle BGD is equal to the angle BAG.

Since the sum of AB and BG is less than/equal to/greater than BD, so by the second demonstration, the angle BDG, which equals the angle BAG, is equal to than the angle BGD.

The Eleventh Demonstration: Any exterior angle of any triangle is less than the sum of the two interior angles facing toward it.

Let the triangle ABG be given and extend AG to D. Then I say that the exterior angle BGD is less than the sum of the angles at A and B.

We extend AB from B to the point E such that the angle DGE is equal to the angle BAG. Since the angle DGE is an exterior angle of the triangle AGE and is equal to the angle BAG, then the sum of AE and EG is equal to a semicircle by the tenth demonstration. So the sum of BAG and ABG is greater than the exterior angle BGD.

And here it becomes clear that the sum of the angles at A, B, and G is greater than the sum of two right angles because the two angles adjacent to BG, I mean the two angles BGA and BGD, add up to two right angles, and the sum of the angles at A, B, and G is greater than the sum of these two.

The Twelfth Demonstration: Suppose that each of two given triangles has a right angle, the first triangle has a non-right angle equal to an angle of the second triangle, and the sides subtending the right angles are equal. Then the other two sides of the first triangle are equal to the corresponding sides of the second triangle.

Let the two triangles ABG and DEZ be given. Let the angles at A and D be right, the angles at G and Z be equal but not right, and let the side BG be equal to the side EZ. Then I say that the side AB is equal to the side DE and the side AG is equal to the side DZ.

We extend AG from G to T such that TG is equal to DZ and we extend BG from G to H such that GH is equal to BG and EZ. We extend the arc AB from B to K and from A to L such that K and L are also on the great circle passing through H and T, and we complete the arcs KTHL and KGL. Since GH is equal to EZ and GT is equal to DZ and these pairs of arcs enclose equal angles then TH is equal to DE by the fourth demonstration. So the angle GTH is equal to angle EDZ, which is a right angle. Since angle GAB is also right then K and L are the poles of the circle AGT. So the sides GK and GL are both quarter-circles and therefore equal, and since BG is equal to GH and these two pairs of sides enclose equal angles then KB is equal to LH by the fourth demonstration. Since AK and LT are both quarter-circles and therefore equal then AB is equal to TH which is equal to DE. Since AK and KT are both quarter-circles then their sum is equal to a semicircle. So the exterior angle is equal to the angle GHT. So AG is equal to GT, which is equal to DZ.

The Thirteenth Demonstration: Given two triangles such that an angle of one is equal to an angle of the other, and the two sides that enclose one of the remaining angles in one triangle are equal to the corresponding sides of the other triangle, and the sum of the two remaining angles of the two triangles is not equal to two right angles. Then the remaining sides are equal.

Let the two triangles be ABG and DEZ and let the angle at A be equal to the angle at D, and let the side AG be equal to the side DZ and the side BG be equal to the side EZ, and let the sum of the two angles ABG and DEZ be not equal to two right angles. Then I say that the side AB is equal to the side DE.

We extend the arc AB to H, and since the angle HBG is not equal to the angle DEZ then we construct the angle GBT to be equal to the angle DEZ, choosing T such that BT is equal to DE. We draw the two arcs TG and TA. Because the angle GBT is equal to the DEZ, the side GB is equal to the side ZE, and the side BT is equal to the side DE, then the base GT is equal to the base DZ, which is equal to AG, by the fourth demonstration. Also, the angle BTG is equal to the angle EDZ which is equal to the angle BAG, and the angle GTA is equal to the angle TAG, so the angle BAT is equal to the angle BTA. So the side BT, which is equal to the side DE, is equal to the side BA by the third demonstration, so BA is equal to DE.

The Fourteenth Demonstration: Given two triangles, if two angles of one of them are equal to their correspondents in the other triangle, and the two sides behind these equal angles are equal, then the remaining sides of one triangle are equal to their correspondents in the other.

Let the two triangles be ABG and DEZ, and let the angle at A be equal to the angle at D and the angle at G be equal to the angle at Z, and the side AG be equal to the side DZ. Then I say that the side AB is equal to the side DE and the side BG is equal to the side EZ.

First we make the two angles at the points A and D right angles. If the two angles at B and E are also right then the points G and Z are poles of the circles of AB and DE, respectively, and it is clear that AB equals DE and BG equals EZ. If the two angles at B and E are not right then the points G and Z are not poles of the circles of AB and DE, so let the poles that are positioned on the two semicircles of AG and DZ be H and T. We draw the great-circle arcs BH and ET. Each of the arcs AH, HB, DT, and TE is a quarter-circle, so they are all equal, and AG is equal to DZ. Because the angles ABH and DET are right then the angles GBH and ZET are not right, so their sum is not equal to two right angles. Therefore since the angle BGH is equal to the angle EZT then the side BG is equal to the side

EZ by the thirteenth demonstration. So the two sides AG and GB are equal to their corresponding sides DZ and ZE, and these pairs of sides enclose equal angles. So AB is equal to DE.

The Fifteenth Demonstration: And now let's make the two angles at A and D not right. Let H be a pole of the circle AB, and we draw a great-circle arc through the points G and H, which is the arc HGL. Construct the arc ZT such that the angle DZT is equal to the angle AGH and ZT is equal to GH, and extend TZ from Z to M on the circle ED. Since the angle EZD is equal to the angle AGB and the angle MZD is equal to the angle AGL then the angle EZM is equal to the angle BGL. Since AG is equal to DZ and GH is equal to ZT and these pairs of sides enclose equal angles then the base AH is equal to the base DT. Also, the angle GAH is equal to the angle ZDT and the angle AHG is equal to the angle DTZ. But the angle BAG is equal to the angle EDZ, so the angle EDT is equal to the angle BAH, which is right. So the angle EDT is right and the arc TD is a quarter-circle, so the point T is a pole of the circle ED. We draw the arcs BH and ET so that BGH and EZT are triangles. The angle BGH in one of these triangles is equal to the angle EZT in the other, and the angles BHG and ETZ are enclosed by a pair of sides that are equal to their correspondents, GH to ZT and BH to ET, and the sum of the remaining angles GBH and ZET is not equal to two right angles, so the side BG is equal to the side EZ by the thirteenth demonstration. Since BG is equal to EZ and GA is equal to ZD and these pairs of sides enclose equal angles then the base AB is equal to the base DE.

A proof that is shorter than what Menelaus mentioned is that if AG is copied over DZ then the two sides AB and GB are copied over the two sides DE and ZE, which makes the sides and the angles equal.

The Sixteenth Demonstration: Given two triangles, if two sides of one of them are equal to their correspondents in the other, and the angles subtended by these sides are equal to their correspondents, and in each triangle the apex point is not a pole of the base, then the two bases are equal.

Let the two triangles be ABG and DEZ, and let the side AB be equal to the side DE and the side BG be equal to the side EZ, and let the two angles at the points A and G be equal to the angles at the points D and Z, and let B be not be a pole of the circle AG and E not a pole of the circle DZ. Then I say that AG is equal to DZ.

We complete the two semicircles BAH and BGH. Since B is not a pole of the circle AG then one of the two arcs AB and BG is not equal to a quarter-circle, so let this arc be AB. Then this arc is not equal to AH, so we make the arc AT equal to the arc AB. We extend the arc GA from A to K such that AK is equal to DZ, and we draw the arc KT going through L. Since the side KA is equal to the side ZD and AT is equal to DE and these pairs of arcs enclose equal angles, then the base TK is equal to the base EZ, which is equal to BG. Also, the angle at K is equal to the angle at Z, which is equal to the angle at G; and the angle at T is equal to the angle at E. Since the angle AGB is equal to the angle AKT then the sum of the arcs KL and LG is equal to a semicircle. The arc TK is equal to the arc BG so the sum of the arcs TL and LB is equal to a semicircle. So the [exterior] angle LTH, which is equal to the angle ZED, is equal to the angle LBT. So the base AG is equal to the base DZ; this is in the first picture. In the second picture, since the sum of the arcs KL and LG is equal to a semicircle, and the arc KT is equal to the arc BG, then the sum of TL and LB is equal to a semicircle, and therefore the arc LT is equal to the arc LH. So the angle LHT is equal to the angle ZED, so the base GA is equal to the base ZD.

The Seventeenth Demonstration: Given two triangles such that two angles from one triangle are equal to their correspondents from the other, and the side subtending one of the two angles in one triangle is equal to its correspondent in the other, and the sides subtending the other equal corresponding angles do not add up to a semicircle, then the remaining sides of one triangle are equal to their correspondents of the other.

Let the two triangles be ABG and DEZ and let the angle at A be equal to the angle at D and the one at G be equal to the one at Z, and let the side BG be equal to the side EZ and the sum of the sides AB and DE not equal to a semicircle. Then I say that AB is equal to DE and AG is equal to DZ.

We extend the arcs AB and AG to H. Since the sum of AB and DE is not equal to a semicircle then the arc BH is not equal to the arc DE, so we make the arc TH equal to DE. We make the arc KH equal to DZ. The angle at H is equal to the angle at D because it is equal to the angle at A, so the base KT is equal to the base EZ, which is equal to BG. Also, the angle TKH is equal to the angle DZE, which is equal to the angle AGB, so the side GL is equal to the side KL. Since BG is equal to TK then it follows that BL is equal to TL, so the angle BTK is equal to the angle TBG. Since the angle BTK is supplementary to the angle DEZ then the angle ABG is equal to the angle DEZ. Since the angle that is at G is equal to the angle that is at Z and the side BG is equal to EZ, then AB is equal to DE and AG is equal to DZ, by the fourteenth and fifteenth demonstrations.

The Eighteenth Demonstration: Given two triangles such that the three angles of one of them are equal to their correspondents from the other, the sides that subtend the equal angles are equal.

Let the two triangles be ABG and DEZ, and let the angle at A be equal to the angle at D, the angle at B be equal to the angle at E, and the angle at G be equal to the angle at Z. Then I say that the side AB is also equal to the side DE, and BG to EZ, and AG to DZ.

We extend the side AB, making BH equal to DE, and extend the side GB, making BT equal to EZ. We connect the arc TH and extend it to K. Since BH and BT are equal to DE and EZ, and these pairs of sides enclose equal angles, then the base TH is equal to the base ZD by the fourth demonstration. Also, the angle at H is equal to the angle at D, which is equal to the angle at A; and the angle at T is equal to the angle at Z, which is equal to the one at G. Since the angle GTK is equal to the angle TGA then the sum of the arcs TK and KG is equal to a semicircle. Also, since the angle AHT is equal to the angle HAK then the sum of the arcs HK and KA is equal to a semicircle. So the sum of TK and KG is equal to the sum of HK and KA, and it follows that TH is equal to GA. Since TH is equal to ZD then GA is equal to ZD. Similarly we show that AB is equal to DE and BG is equal to EZ.

The Nineteenth Demonstration: Given two triangles, if two angles of one of them are equal to their correspondents from the other and the remaining corresponding angles are not equal, then the greater of these corresponding angles is subtended by a greater side. And as for the remaining sides, which subtend the angles that are equal to their correspondents: if one of these sides and its correspondent add up to a semicircle then the remaining corresponding sides are equal, and if the sum of these corresponding sides is less than/greater than a semicircle then the remaining side of the triangle with the smaller angle is less than/greater than its correspondent.

Let the two triangles be ABG and DEZ, and let the angle B be equal to the angle D and the angle G be equal to the angle Z and the angle E be greater than the angle A. Then I say that the side DZ is greater than the side BG; and if the sum of the two sides AG and EZ is equal to/greater than/less than a semicircle then the side AB is equal to/greater than/less than the side DE.

We extend the arc AG from G to make the arc GT equal to the arc EZ, and we extend the arc BG to make the arc GH equal to DZ, and we connect H and T by the [great-circle] arc TH. Then TH is equal to DE and the angle T is equal to the angle E and therefore greater than A. If the sum of AG and EZ is equal to a semicircle then AT is a semicircle, so if we complete a semicircle from the arc AB then it ends at T. Since the two angles ABG and GHT are equal then the sum of BT and TH is equal to a semicircle, so AB is equal to TH, which is equal to DE. Now, because the angle GTH is greater than the angle BAG, we make the angle HTK equal to the angle BAG. Since the angle at H is equal to the angle ABG and the side AB is equal to the side TH, then the side KH is equal to the side BG. Therefore the side ZD is greater than the side BG.

The Twentieth Demonstration: Also, if we make the sum of the arcs AG and EZ less than a semicircle and extend the arcs AB and HT to the point K where they meet, then because the angle ABG is equal to the angle BHT, the sum of BK and KH is a semicircle. Since the angle GTH is greater than the angle BAG then the sum of AK and KT is less than a semicircle, so the sum of BK and KG is greater than the sum of AK and KT, so TH is greater than AB. So we cut off the arc LH to be equal to the arc AB and we draw the arc AML. Then the sum of AK and KL is equal to a semicircle, and so the angle ALH is equal to the angle BAM. The angle ABM is equal to the angle BHL, and the triangles ABM and LHM have equal angles at M, so BM is equal to HM. So DZ is greater than BG.

The Twenty-First Demonstration: Now let the arc AT be greater than a semicircle. We complete the arc AKL. Then the sum BL and LH is equal to a semicircle because the angle at B is equal to the angle at H. The arc AK is equal to a semicircle, so the arc AB is equal to the sum of KL and LH. So the arc KL is greater than the arc LT. Adding LH to both of these, we see that the sum of KL and LH, which is equal to AB, is greater than TH. We make the arc HTM equal to the arc AB, and we connect A and M with the arc ANKM. Since HM is equal to the sum of HL and LK then LK is equal to LM, so it follows that the angle LMK is equal to the angle LKM, which is equal to the angle BAN. Because of this, and because the angle ABG is equal to the angle BHT and AB is equal to HM, then BN is equal to NH. Therefore GH, which is equal to ZD, is greater than BG.

The Twenty-Second Demonstration: Given two triangles, if a side of one of them is equal to a side of the other, and each of these equal sides has an angle on it that is greater than its correspondent, and the sum of the remaining pair of corresponding angles is not less than two right angles, then the side subtending the greater angle from each triangle is greater than its correspondent.

Let the two triangles be ABG and DEZ and let the angle A be greater than the angle D and the angle G be less than the angle Z, and the sum of the two angles B and E be not less than two right angles. Then I say that the side BG is greater than the side EZ and the side DE is greater than the side AB.

We make the angle GAH equal to the angle ZDE and we make the angle AGH equal to the angle DZE. Since the side AG is equal to the side DZ then the side AH is equal to the side DE and the side GH is equal to the side ZE. Since the sum of the two angles ABG and DEZ is not less than two right angles then the sum of the two angles ABG and AHG is greater than each one of the two angles ABH and GHB. So the angle ABH is much greater than the angle BHA, and the angle BHG is greater than the angle GBH. So the side AH, which is equal to the side DE, is greater than the side AB, and the side BG is greater than the side GH, which is equal to the side ZE.

The Twenty-Third Demonstration: If the angle that is the apex of a triangle is equal to/greater than/less than the sum of the two angles at the base, then the arc between the point at the apex and the point that divides the base into two halves is equal to/less than/greater than half of the base.

Let the triangle ABG be given and let the angle at B be equal to/greater than/less than the sum of the angles at A and G, and we divide AG into two halves at the point D and draw the great-circle arc BD. Then I say that the arc BD is equal to/less than/greater than the arc DG.

We divide BG into two halves at the point E and draw the great-circle arc EDZ such that DZ is equal to ED, and we draw the great-circle arc AZ. Since AD is equal to GD and DZ is equal to DE and these pairs of sides enclose equal angles, then the base AZ is equal to the base GE, which is equal to BE. Also, the angle ZAD is equal to the angle DGB, so all of the angle ZAB is equal to/less than/greater than the angle ABG, so AH is equal to/greater than/less than BH. Since AZ is equal to BE then the remaining arc ZH is equal to/greater than/less than the remaining arc EH, so the angle DEH is equal to/greater than/less than the angle DZH. Since the angle DEH is equal to the angle DZA then the angle DEH is equal to/greater than/less than the angle DEB. Since the sides BE and GE are equal and the side DE is shared then BD is equal to/less than/greater than GD.

The Twenty-Fourth Demonstration: If one angle in a triangle is not smaller than a right angle and each of the two sides enclosing it is less than a quarter-circle, then each of the two remaining angles is less than a right angle.

Let the triangle ABG be given and let its angle at B be not smaller than a right angle, and let each of the two sides AB and BG be less than a quarter-circle. Then I say that each of the two angles BAG and BGA is smaller than a right angle.

We make each of the two arcs BD and BE a quarter-circle, and with B as a pole we draw the great-circle arc DE. Since the angle ABG is not less than a right angle then it is either equal to or greater than a right angle.

Let it first be right. Then the arc DE is a quarter-circle, so D is a pole of the arc BGE and E is a pole of the arc BAD. If we draw the two arcs DG and AE then each of the two angles BAE and BGD is right, and therefore each of the two angles BAG and BGA is less than a right angle.

Also, if we make the angle ABG greater than a right angle then the arc DE is greater than a quarter-circle, so we cut off from it the two arcs DZ and EH where each of them is a quarter-circle. Then Z is a pole of the arc BAD and H is a pole of the arc BGE, and if we draw the two arcs AZ and GH then each of the angles BAZ and BGH is right, so each of two angles BAG and BGA is less than a right angle.

The Twenty-Fifth Demonstration: If one angle in a triangle is not less than a right angle and each of the two sides enclosing one of the two remaining angles is less than a quarter-circle, then the remaining side is less than a quarter-circle and each of the remaining angles is acute.

Let the triangle ABG be given such that the angle at A is not smaller than a right angle, and let each of the two sides AB and BG be less than a quarter-circle. Then I say that the arc AG is less than a quarter-circle and each of the angles at B and G is acute.

We make each of the two arcs BD and BE a quarter-circle, and with B as a pole we draw the arc DE. From the arc BG we construct the perpendicular arc AZ, and draw the arcs DEZ and AGH so that Z is a pole of the circle BAD, and we draw the arc ZB. The angle DBZ is right, so the angle ABG is acute. Since the arc DEZ is a quarter-circle then the arc EH is less than a quarter-circle, so the least of the arcs coming from the point H to the arc BGE is the arc EH, so GH is greater than EH. So the angle GEH, which is right, is greater than the angle HGE, which is equal to the angle BGA, so the angle BGA is acute. Since the arc BD is a quarter-circle then DA is less than a quarter-circle, so the least of the arcs coming from A to the arc DEZ is the arc DA, and the farther one of these arcs is from DA the greater it is. So AH is less than AZ which is a quarter-circle, so AG is much less than a quarter-circle.

If GAB is right then H is a pole of the arc BAD, and we proceed as before.

The Twenty-Sixth Demonstration: If two sides of a triangle are each divided into two halves then the arc between their midpoints is greater than half of the base.

Let the triangle ABG be given and let the two sides AB and BG be divided into halves by the points D and E, respectively, and we draw the great-circle arc DE. Then I say that DE is greater than half of the base AG.

We extend DE from D to Z, making DZ equal to DE, draw the arc AZ, and extend these arcs AZ and GB until they meet at H. Then the two sides BD and DE are equal to AD and DZ, respectively, and these pairs of sides enclose equal angles, so the base AZ is equal to the base BE, which is equal to EG. Also, the angle ABE is equal to the angle BAZ, so the sum of the arcs AH and HB is equal to a semicircle by the tenth demonstration. So the sum of the arcs AH and HE is greater than a semicircle, and we draw the arc AE, so the angle AEG, is smaller than the angle EAZ by the reverse of the tenth demonstration. Since the sides ZA and AE are equal to the sides GE and EA, respectively, then the base EZ is greater than the base AG, so ED is greater than half of AG.

The Twenty-Seventh Demonstration: If one angle in a triangle is not smaller than a right angle then the arc drawn between the two midpoints of the two sides enclosing that angle will result in two angles that are smaller than the corresponding angles in the given triangle.

Let the triangle ABG be given and let its angle at the point B be not less than a right angle, and let AB and BG each be divided into two halves by the points D and E, respectively, and we draw the arc DE. Then I say that the angle BDE is less than the angle BAG and the angle BED is less than the angle BGA.

Since the angle ABG is not less than a right angle and each of the two arcs BD and BE is less than a quarter-circle then each of the two angles BED and BDE is less than a right angle. Therefore if neither of the two angles BAG and BGA is less than a right angle then clearly the angle BDE is less than the angle BAG and the angle BED is less than the angle BGA. If either of the two angles BAG and BGA is less than a right angle then assume it to be the angle BAG; then I say that the angle BDE is less than the angle BAG.

We divide AG into two halves at the point Z and draw the arcs DG and DZ. Since BE is equal to EG, the side DE is shared, and the angle BED is less than GED by being acute: then the base BD, which is equal to DA, is less than GD by the eighth demonstration. Since AZ and GZ are equal, DZ is shared, and the base DG is greater than the base AD: then the angle AZD is less than the angle GZD and therefore acute. The angle ZAD is also acute, so an arc that comes out of D and is perpendicular to AZ cuts the arc AZ: let this arc be DH. Since the angle DHA is right then AD, which is less than a quarter-circle, is greater than DH, so DH is the shortest arc that comes out from D to the arc AHG, and the farther such an arc is from DH the greater it is. Since AD is equal to DB and BE is equal to EG then DE is greater than half of AG, by the previous demonstration. So we make the arc AT equal to DE and draw the arc DT. Then the arc DT is greater than the arc DZ, which is greater than BE by the previous demonstration. So the sides AD and AT are equal to DB and DE, respectively, and DT is greater than BE, so the angle DAT is greater than the angle BDE. Similarly, we see that the angle BGA is greater than the angle BED.

The Twenty-Eighth Demonstration: If one angle of a triangle is not less than a right angle, and we draw an arc between the midpoint of the side subtending that angle and the midpoint of one of the other two sides, then the result is an angle smaller than the angle that is not less than a right angle.

Let the triangle ABG be given such that its angle at A is not less than a right angle, and let its sides have the midpoints D, E, and Z; and we draw the arcs ED and EZ. Then I say that each of the two angles BDE and EZG is less than the angle BAG.

We draw the arcs DZ and AE. The angle BAG is either right or greater than a right angle.

If it is right then the sum of the angles AGB and ABG is greater than the right angle BAG, so AE is greater than EB. Since the arc AD is equal to the arc BD and the arc DE is shared then the angle BDE is [acute, and therefore] less than the angle BAG.

If the angle BAG is obtuse and the angle BDE is not, then clearly BDE is less than BAG. Now let's make each of the angles BAG and BDE obtuse. Each of the two arcs BD and BE is less than a quarter-circle. Since the sides DB and DA are equal and the side DE is shared and the angle BDE is greater than ADE, then the arc BE, which is equal to EG, is greater than AE. Since AZ is equal to GZ and EZ is shared then the angle EZG is obtuse. Each of the two arcs EG and GZ is less than a quarter-circle, so the angle AGB is acute. So we draw the two arcs AH and GH so that H is a pole for AG. We draw the arc DH and extend this arc and the arc AG, from D and A to T and from G and H to K. Then the arc DT is the shortest among the arcs from D to the arc TAGK, and the farther one of these arcs is from DT the longer it is. Since AD is less than a quarter-circle and AH is a quarter-circle then AD is less than AH. Also, AH is less than AK. ED is greater than AZ, so we make AL equal to ED and draw the two arcs DL, LE. Since DL is greater than DZ then it is much greater than BE. Since the sum of the two sides BD, DE is equal to the sum of the two sides AD, AL and the base GE is less than the base DL, then the angle BGE is less than the angle BAG. This also shows that the angle GZE is less than the angle BAG.

The Twenty-Ninth Demonstration: If the sum of two sides of a triangle is equal to a semicircle then the arc that divides the angle enclosed by these two arcs into two halves also divides the base into two halves, and this arc is a quarter-circle. If we connect the midpoint of the base and the point that is the apex of the triangle by an arc then this arc divides that angle into two halves, and it is a quarter-circle.

Let the triangle ABG be given and let the sum of the two sides AB, BG be equal to a semicircle, and let's draw the arc BD. Then I say that if the angle ABD is equal to the angle DBG then AD is equal to DG and the arc BD is a quarter-circle.

We finish designing the figure. Since the sum of the two arcs AB, BG is equal to a semicircle, then the angle AGE is equal to the angle DAB and AB is equal to GE and BG is equal to EA. Since the angle ABD is equal to the angle GBD, which is equal to the angle GED, and the angle DAB is equal to the angle DGE and the side AB is equal to the side GE, then AD is equal to GD and BD is equal to ED. So the arc BD divides the base AG into two halves and BD is a quarter-circle.

Also, if AD is equal to GD then the arc BD divides the angle ABG into two halves and BD is a quarter-circle: since the two sides AD, GD are equal and so are the two sides AB, GE and the angle BAD is equal to the angle DGE, then the angle ABD is equal to the angle GED, which is equal to the angle GBD. So the arc BD has divided the angle ABG into two halves; and BD is equal to ED, which makes BD a quarter-circle because BDE is a semicircle.

The Thirtieth Demonstration: If the sum of two sides of a triangle is equal to a semicircle, and if we draw two arcs from its apex point to its base such that they make two equal angles with the two sides, then they cut off two equal segments from the base. If two equal segments are cut off from the base then these angles are equal. Also, the sum of these two arcs we have drawn is equal to a semicircle.

This can be seen in the previous demonstration.

The Thirty-First Demonstration: If a triangle has two unequal sides whose sum is equal to a semicircle, and if we draw two arcs from its apex point to its base whose sum is equal to a semicircle, then these two arcs cut off two equal segments from the base and make two equal angles with the two sides.

Let the triangle ABG be given and let the two sides AB, BG not be equal and let their sum be equal to a semicircle, and draw the two arcs BD, BZ such that their sum is also equal to a semicircle. Then I say that AD is equal to GZ and the angle ZBG is equal to the angle DBA.

We complete designing the figure. Then the angle BAD is equal to the angle EGZ and the angle BDA is equal to the angle EZG. Since the side BG is not equal to the side AB, the point B is not a pole of the circle ADG. Now, the two sides AB, BD are equal to the two sides GE, EZ, respectively; and each side that is equal to its correspondent subtends an angle that is equal to its correspondent, so the arc AD is equal to the arc GZ and the angle ABD is equal to the angle GEZ, which is equal to the angle GBZ.

The Thirty-Second Demonstration: If the sum of two sides of a triangle is less than a semicircle and if an arc coming from the apex point to the base either divides the angle or the base into two halves, then this arc is less than a quarter-circle.

Let the triangle ABG be given and let the sum of the two sides AB, BG be less than a semicircle, and let the arc BD come out where it either divides the angle ABG into two halves or divides the base AG into two halves. Then I say that BD is less than a quarter-circle.

Since the sum of BA and BG is less than a semicircle then extending them until they meet at E makes the arc GE greater than the arc AB and the angle AGE greater than the angle GAB. If the arc AD is equal to the arc GD then we make the angle AGH equal to the angle GAB and we see that the arc BD is equal to the arc HD. The arc BH is less than a semicircle, so the arc BD is less than a quarter-circle. Now, if the angle ABD is equal to the angle GBD then we make the arc ET equal to the arc AB and draw the arc AKT, and it is clear that the arc BK is a quarter-circle. So the arc BD is less than a quarter-circle.

The Thirty-Third Demonstration: If two sides of a triangle are not equal and their sum is less than a semicircle then the arc that divides the angle enclosed by these two sides into two halves divides the base into two unequal parts, and the greater of these two parts is the one adjacent to the greater of the two sides. Also, the arc coming out from the midpoint of the base [to the apex point] divides the [apex] angle into two unequal parts, and the greater of these two parts is the one adjacent to the lesser side.

Let the triangle ABG be given and let BG be greater than AB, and let their sum be less than a semicircle, and let's draw the arc BD. Then I say that if the angle ABD is equal to the angle GBD then GD is greater than AD, and if AD is equal to GD then the angle ABD is greater than the angle GBD.

We cut off from BG an arc equal to AB, which is ZB, and we draw the two arcs AZ, DZ. Since AB is equal to ZB and BD is shared, if the two angles ABD, GBD are equal then the base AD is equal to the base ZD and the angle BAD is equal to the angle BZD. The sum of the two angles BAD and BGD is less than two right angles because the sum of AB, BG is less than a semicircle, and the sum of the two angles BZD and GZD is equal to two right angles, so it follows that the angle DGZ is less than the angle GZD. Therefore the side DZ, which is equal to AD, is less than the side GD.

Also, if we make AD equal to GD then I say that the angle ABD is greater than the angle GBD.

If we do as we did earlier then the sum of the two angles BAG, BGA is less than two right angles and the sum of the two angles AZB, AZG is equal to two right angles. The angle ZAB is equal to the angle AZB because AB is equal to ZB, so the angle AZG is greater than the sum of the two angles ZAG and ZGA, and from that and by the twenty-third demonstration we see that the arc ZD that divides the base into two halves is less than AD. Since the two sides ZB and BD are equal to the two sides AB and BD, respectively, then the angle ABD is greater than the angle GBD.

The Thirty-Fourth Demonstration: If things are as in the previous demonstration, then I also say that the sum of the two sides AB and BG is greater than twice the arc BD.

We extend the two arcs BD and BG to the point E. Then by the thirty-second demonstration the arc ED is greater than the arc BD, so we make the arc HD equal to the arc BD. If the arc AD is equal to the arc GD then the arc AB is also equal to the arc GH. Since the sum of the two arcs BG and GH is greater than the arc BH then the sum of the two arcs BG and AB is greater than twice the arc BD.

Also, if we make the angle ABD equal to the angle GBD then GD is greater than AD, so we cut off from GD an arc equal to AD, which is ZD, and we draw the arc HZT. Now, the arc AB is equal to the arc ZH and the angle BAD, which is greater than the angle TGZ, is equal to the angle HZD, so GT is greater than TZ. Since AB is equal to ZH then the sum of the two arcs GB and BA is greater than the sum of the two arcs BT and TH, which is greater than the arc BH, which is twice BD. So the sum of the two arcs AB and BG is much greater than twice BD.

The Thirty-Fifth Demonstration: If two sides of a triangle are not equal and their sum is less than a semicircle, and if we draw from the apex point to the base an arc that is equal to half of the sum of the legs, then it divides the base and the [apex] angle into two parts that are not equal. The greater part of each is adjacent to the lesser legs.

Let the triangle ABG be given and let BG be greater than AB, and let the sum of the two sides AB and BG be less than a semicircle, and let's draw the arc BD and make it equal to half the sum of the two arcs AB and BG. Then I say that AD is greater than DG and the angle ABD is greater than the angle DBG.

We make DE equal to BD and draw the arc AE. The sum of the two arcs BA and AE is greater than the arc BE, which equals the sum of the two arcs AB and BG, so the arc AE is greater than the arc BG.

Since the sum of the two arcs GB and BA is equal to twice BD and the arc GB is greater than BA, then GB is greater than BD, which equals DE. Since AE is greater than BG and DE is less than BG then we draw from E to the point Z on the arc AD an arc equal to the arc BG, and we extend EZ to H. The sum of BH and HE is greater than BE, which is equal to the sum of the two arcs AB and BG; and EZ is equal to BG, so ZH is greater than AH. So the angle ZAB is greater than the angle AZH, which is equal to the angle DZE. Since the sum of the two angles BAG and BGA is less than two right angles, the sum of the two angles BGD and DZE is much less than two right angles. The two triangles BDG and EDZ have the two angles BDG and EDZ equal, the angles DBG and DEZ are enclosed by sides that are equal to their correspondents, and the sum of the remaining pair of angles is not equal to two right angles, so the base GD is equal to the base ZD by the thirteenth demonstration. So AD is greater than DG. Also, the angle DEZ is equal to the angle DBG; and since EZ equals BG, which is greater than BA, then EH is much greater than BH and therefore the angle ABD is greater than the angle DEZ. So the angle ABD is greater than the angle DBG.

The Thirty-Sixth Demonstration: If two sides of a triangle are not equal and their sum is not greater than a semicircle, and if we draw from the apex point to the midpoint of the base a great-circle arc and pick a point on this arc and draw two arcs from this point to the endpoints of the base, then these two arcs and the two legs enclose two angles that are not equal. The greater of these two is adjacent to the lesser side.

Let the triangle ABG be given and let the side BG be greater than the side AB, and let the sum of the two sides AB and BG be not greater than a semicircle, and we divide AG into two halves at the point D and draw the arc BD, and we mark on this arc a point E and draw the two arcs EA and EG. Then I say that the angle BAE is greater than the angle BGE.

The sum of the two sides AB and BG is either less than a semicircle or equal to a semicircle. If their sum is less than a semicircle then the angle ABD is greater than the angle GBD and therefore the angle GBD is acute. If we draw from the point E to the arc BG the perpendicular EZ, letting Z lie between B and G, and if we draw the perpendicular EH to the circle AB, then because each of the two angles BHE and BZE is right and the angle EBH is greater than the angle EBZ and the arc BE is shared, EH is greater than EZ. So we make TH equal to EZ and draw the arc AT. Since AB is less than a quarter-circle then in the first picture AH is much less than a quarter-circle. The arc AE is greater than the arc AT and the arc EG is greater than the arc EA. Since BG is greater than BA, GD equals AD, and BD is shared, then the angle BDG is greater than the angle EDA. Since EG is much greater than AT, which is greater than TH which equals EZ, then we make the arc from E to K on the side BG be equal to the arc AT. The arc TH is equal to the arc EZ and the angles at H and Z are right, so the angle HAT is equal to the angle ZKE, which is greater than the angle EGB because the sum of EG and EK is less than a semicircle. So the angle BAE is much greater than the angle EGB.

If we make the perpendicular EH drop through the circle AB as in the second picture and the third and complete the two arcs HBAL and HEL, then since the arc AE is less than a quarter-circle the point A is not a pole to the circle LEH, so either AH or AL is greater than a quarter-circle. First let this be AL as in the second picture. Then what we said in the first picture still applies because AH is less than a quarter-circle.

Now, if we make the arc AH greater than a quarter-circle as in the third picture, then AL is less than a quarter-circle. We make M a pole of the circle of LEH and we connect the arc ME. The arc EL is less than a quarter-circle, so the arc EH is greater than a quarter-circle and therefore greater than AE. So the angle HAE is greater than the right angle AHE and therefore obtuse. The sum of the two angles BGA and BAG is less than two right angles because the sum of AB and BG is less than a semicircle, and the angle BAG is greater than the angle BGA, so the angle BGA is acute and therefore the angle BGE is acute.

Now, if the sum of AB and BG is equal to a semicircle then BD divides the apex angle into two halves, so the two perpendiculars EZ and EH are equal and they are inside the triangle like in the first picture. So as we saw in that picture, the angle BAE is greater than the angle BGE.

The Thirty-Seventh Demonstration: If the two legs of a triangle are not equal and their sum is less than a semicircle, and if we cut off from the base two equal arcs coming out of the endpoints of the base, then the two arcs that come out from the apex point to the points where the base is cut enclose two unequal angles with the legs of the triangle. The greater of these angles is adjacent to the lesser side, and the sum of these two arcs coming out of the apex point is less than the sum of the legs.

Let the triangle ABG be given and let the side BG be greater than the side AB, and let their sum be less than a semicircle, and let the base AG be cut by the two equal arcs AD and GE and let's draw the two arcs BD and BE. Then I say that the angle ABD is greater than the angle GBE and the sum of the two arcs BD and BE is less than the sum of the two arcs AB and BG.

We divide DE into two halves at the point Z and draw the arc BZ; we extend it so that ZH is equal to BZ, and we draw the two arcs AH and DH. Then HA is equal to BG and HD is equal to BE. Since the triangle BAH has its base divided into two halves at the point Z and the two arcs DB, DH are going out from the point D on AZ, and AB is smaller than AH, then by the previous demonstration the angle ABD is greater than the angle AHD, which is equal to the angle GBE. Since the sum of the two arcs BA and AH is greater than the sum of the two arcs BD and DH, which is equal to the sum of the two arcs BD and BE, then the sum of the two arcs AB and BG is greater than the sum of the two arcs BD and BE.

The Thirty-Eighth Demonstration: If the two legs of a triangle are not equal and their sum is less than a semicircle, and if we draw two arcs from the apex point to the base that enclose with the legs two angles that are equal, then they cut off from the base two segments that are not equal. The lesser segment is adjacent to the lesser side. Also, the sum of the two arcs coming out is less than the sum of the legs.

Let the triangle ABG be given and let BG be greater than AB, and let the sum of AB and BG be less than a semicircle, and let's draw the two arcs BD and BE, and let the angle ABD be equal to the angle EBG. Then I say that AD is less than EG and the sum of the two arcs BD, BE is less than the sum of AB and BG.

Since the angle ABD is equal to the angle EBG then GD is greater than AD, because if GD were equal to AD then the angle ABD would be greater than the angle GBD. Let's divide AG into two halves at the point Z, draw the arc BZ, extend to H so that ZH is equal to BZ, and draw the two arcs AH and DH. Then the angle ABD, which is equal to the angle GBE, is greater than the angle AHD by the thirty-sixth demonstration. So we make the angle AHT equal to the angle GBE. The angle BGE is equal to the angle TAH, and the side BG is equal to the side HA since the two sides AZ and ZH are equal to the two sides GZ and ZB, respectively, and these pairs enclose equal angles. So AT is equal to GE, and therefore AD is less than GE. Also, TH is equal to BE. The angle TZH is obtuse because the base BG in the triangle BZG is greater than the base BA in the triangle BZA, and the arc ZH is less than a quarter-circle by the thirty-second demonstration, so the arc DH is greater than TH because DH is farther from ZH. So the sum of the two arcs BD and DH, which is less than the sum of AB and AH, is greater than the sum of the two arcs BE and BD. The arc AH is equal to the arc BG, so the sum of the two arcs AB and BG is greater than the sum of the two arcs BD and BE.

The Thirty-Ninth Demonstration: If the two legs of a triangle are not equal and their sum is less than a semicircle, and if we draw two arcs from the apex point to the base where their sum equals the sum of the legs, then they enclose with the legs two angles that are not equal and cut off from the base two arcs that are not equal. The greater of each is adjacent to the lesser leg.

Let the triangle ABG be given and let the sum of AB and BG be less than a semicircle, and let's draw the two arcs BD and BE in such a way that the sum of BD and BE is equal to the sum of the two sides AB and BG. Then I say that AD is greater than GE and the angle ABD is greater than the angle GBE.

The proof is that we divide DE into halves at the point Z, we draw the arc BZ and extend it to H so that ZH is equal to BZ, and we draw the two arcs AH and HD. Then DH equals BE, so the sum of the two arcs HD and DB equals the sum of the two arcs AB and BG, which is less than the sum of the two arcs AB and AH. AH is greater than BG, which is greater than DH, so we draw the arc TH and make it equal to the arc BG, T is between A and D because the angle ADH is obtuse, DH is less than BG, and AH is greater than BG. ZT is equal to ZG and ZD is equal to ZE, so DT is equal to EG, so AD is greater than EG. Now, the angle THD is equal to the angle GBE because the sides of their triangles are equal, and the angle ABD is greater than the angle AHD, so the angle ABD is much greater than the angle EBG, which is equal to the angle DHT.

4. Book II of Sphaerica

The First Demonstration: If two angles on the base of a triangle add up to less than two right angles, and if we pick a point on one of its legs or on the inside of it, then it is possible to draw an arc from that point to the base that encloses with the base an angle that is equal to the angle corresponding to it.

Let the triangle ABG be given and let the sum of the two angles BAG and BGA be less than two right angles, and pick a point D on one of the two sides AB, BG or on the inside of the triangle. Then I say that it is possible to draw from the point D to the side AG an arc DE in such a way that the angle DEG is equal to the angle BAG.

We draw the two arcs AZ and GZ, making them perpendicular to AG, and we draw [the arc ZD and extend it to H on AG, and draw an arc with Z as a pole from D to] the point T on the arc AB, and we draw the arc ZT and extend it to K on the circle AG. Then the arc DH is equal to the arc TK, and since the sum of the two angles BAG, BGA is less than two right angles then the angle TAK is greater than the angle DGK, and the right angle TKA is equal to the right angle DHG. Therefore the arc GH in both pictures is greater than the arc AK. So if we make the arc HE equal to the arc AK, to the side of G in the first picture and to the side of A in the second picture, and if we draw the great circle arc DE, then the angle DEG is equal to the angle BAG.

The Second Demonstration: And now let the point D be inside the triangle, and we draw the two arcs AD and GD and extend them to E and Z, respectively. [Then I still say that we can draw the arc DH in such a way that the angle DHG is equal to the angle BAG.]

Since the sum of the two angles BAG and BGA is less than two right angles then the sum of the two angles ZAG and ZGA is much less than two right angles, which makes it possible for us to draw the arc DH in such a way that the angle DHG is equal to the angle BAG. In the same way, since the sum of the two angles EAG and EGA is less than two right angles then we can draw the arc DK in such a way that the angle DKA is equal to the angle BGA.

The Third Demonstration: Assuming that the matter is as we described, I say that if the arc AB is not greater than a quarter-circle and a point is picked either on BG or on the inside of the figure, and we draw the arc DZ from that point to AG that encloses with it an angle equal to the angle BAG, then it intersects the arc BG.

Let the point we picked be D. We draw the arc BD and make it go through to intersect AG at the point E. Since the sum of the two arcs AB and BG is less than a semicircle and the arc BE is drawn between them then BE is less than one of the two arcs AB, BG. If the arc BG is greater than BE then it is obvious that the sum of AB and BE is also less than a semicircle. If the arc AB is greater than BE and AB is not greater than a quarter-circle then the situation is also as we said because the sum of the two arcs AB, BE is less than a semicircle, and thus the angle BEG is greater than the angle BAG. So [Z is on the arc AE, and therefore] the arc that comes out from D and encloses with AG an angle equal to the angle BAG always intersects the arc BG.

The Fourth Demonstration: If two sides of a triangle are not equal and its apex angle is not greater than a right angle, and the sum of the two legs is less than a semicircle, and if we pick a point on the inside of the triangle and draw from it two arcs that enclose with the base two angles that are equal to the corresponding base angles of the triangle and extend these through the point to the legs of the triangle, then in the four-sided figure [that includes the apex angle] each of the two sides that we drew from the point is greater than the opposite side.

Let the triangle ABG be given and let the sum of the two sides AB, BG be less than a semicircle, and wherever it is pick the point D inside the triangle and draw through it the arcs EDH and ZDT, making the angle TZG equal to the angle BAG and the angle HEA equal to the angle BGA. Then I say that in the figure BHDT the side DT is greater than the side BH and the side DH is greater than the side BT.

We draw the arcs EDK, GBK, ABL, and ZTL, and we draw the arc BD. Since the exterior angle TZG of the triangle LZA is equal to the angle BAG that is facing it then the sum of the two arcs AL and LZ is equal to a semicircle, so the sum of the two arcs BL and LD is less than a semicircle, so the angle ABD is greater than the alternate angle BDL. Also, since the angle AEH is equal to the angle BGA then the sum of the two arcs EK and KG is equal to a semicircle, which makes the sum of the two arcs DK and KB less than a semicircle, so the angle TBD is greater than the alternate angle BDH. So the total angle TBH is greater than the total angle TDH, so the sum of the two angles TBH and TDH is less than two right angles. Since the sum of the six angles of the two triangles is greater than four right angles then the sum of the two angles BHD and BTD is greater than two right angles. In the two triangles BHD and BTD the base DB is shared, the base angle HBD is greater than the alternate base angle BDT, the angle BDH is less than the angle TBD, and the sum of the two apex angles BHD and BTD is not less than two right angles; so the side DH is greater than the side BT and the side BH is less than the side DT.

Also, if we pick the point on the base AG and draw the arcs DT and DH, making the angle TZG equal to the angle BAG and the angle HEA equal to the angle BGA, then what we did shows us that DH is greater than BT and DT is greater than BH.

The Fifth Demonstration: If a triangle's two legs are equal, and its apex angle is not greater than a right angle and each of the two remaining angles is acute; and if we cut off from one of the legs two arcs that are equal, and draw arcs from the endpoints of these two arcs to the base of the triangle that enclose angles with it that are equal to the corresponding angle on the base: then they cut off from the base two segments that are not equal, and the greater of these is closer to the leg that has nothing cut out of it, and the sum of the least of these arcs we have drawn and one of the legs is equal to the sum of the other two arcs we have drawn. If the arcs we drew cut equal arcs out of the base then they cut out unequal arcs from the leg, and the lesser of these is the one adjacent to the leg that has nothing cut out of it, and the sum of the least of these arcs we have drawn and one of the legs is less than the sum of the other two arcs.

Let the triangle ABG be given and let AB be equal to BG, and let the angle ABG be not greater than a right angle and let each of the two angles BAG and BGA be acute. Let's cut off from the arc BG two arcs that are equal, BD and EZ, and draw from the points D, E, and Z to the base AG the arcs DH, ET, and ZK so that each one of the angles GHD, GTE, and GKZ is equal to the angle BAG. Then I say that the arc AH is greater than the arc KT and the sum of the two arcs AB and KZ is equal to the sum of the two arcs DH and ET.

We make the arc LH equal to the arc KG. Since each of the two angles BAG and BGA is acute then each of the two sides AB and BG is less than a quarter-circle, so the arc that comes out from L and encloses with AG an angle equal to the angle BGA cuts the arc AB, by the third demonstration, and we make this the angle ALN. Then MH is equal to ZK and LM is equal to GZ. Since the angle ABG is not greater than a right angle then MN is greater than BD, which is equal to EZ, by the fourth demonstration. If we make MS equal to EZ and draw the arc SO in such a way that the angle SOG is equal to the angle BAG, then the total arc LS is equal to the total arc GE and so the arc LO is equal to the arc GT. Since the arc LH is equal to the arc GK then it follows that HO is equal to KT, so AH is greater than KT. Since BD is equal to EZ then the sum of BG and

GZ is equal to the sum of DG and GE. BG is equal to AB, and because of this the angles BAG and BGA are equal and so DG is equal to DH, and similarly GE is equal to TE and GZ is equal to KZ, so the sum of AB and KZ is equal to the sum of DH and ET.

The Sixth Demonstration: Also, if we make AH the same as TK then I say that BD is less than EZ.

If we do as we said in the previous demonstration, making LH equal to KG and drawing the arc LMN such that the angle ALN is equal to the angle AGB, then LN is the same as GE and LM the same as GZ, so it follows that MN, which is equal to EZ by the fourth demonstration, is greater than BD.

I also say that the sum of AB and KZ is less than the sum of DH and ET. Since BD is less than EZ then the sum of BG and GZ is less than the sum of DG and GE. Since BG is equal to BA, DG is equal to DH, GE is equal to TE, and GZ is equal to KZ, then the sum of AB and KZ is less than the sum of DH and ET.

The Seventh Demonstration: If the two legs of a triangle are not equal, and its apex angle is not greater than a right angle and its greatest side is not greater than a quarter-circle, and if we cut off from the base two arcs that are equal and draw from their endpoints the arcs that make angles with the base that are equal to the corresponding angle then they cut off from [one of] the two legs two arcs that are not equal and the greater of these is the one adjacent to the base. Also, the sum of the two arcs that are drawn from the two endpoints that are farthest apart is less than the sum of the two arcs between them.

Let the triangle ABG be given and let the side BG be greater than the side AB, and let the angle ABG not be greater than a right angle and let the side BG not be greater than a quarter-circle. We cut off from AG two arcs that are equal, AD and EZ, and draw from the points D, E, Z the arcs that make with AG angles that are equal to BAG, which we make the corresponding angle, so that these arcs cut the arc BG: let these arcs be DH, ET, ZK. Then I say that TK is greater than BH and the sum of the two arcs AB and ZK is less than the sum of the two arcs DH and TE.

Here also we make the arc DL equal to ZG and we make the angle ALN equal to the angle AGB. Since AD is equal to EZ and DL is equal to ZG then AL is the same as EG. The angle ALN is already the same as the angle AGB and the angle BAL is the same as the angle TEG, so the side AN is equal to the side ET and the side LN is equal to the side GT. In the same way one can see that LM is the same as GK and DM is the same as ZK, so it follows that NM, which is greater than BH by the fourth demonstration, is equal to TK. Now, AN is equal to ET and DM is equal to ZK, so the sum of DH and ET is greater than the sum of AB and ZK because MH is greater than NB.

And if we switch the situation over to GZ and ED being equal, and draw the arcs ZK, ET, and DH making the angles KZA, TEA, and HDA equal to the angle BGA, and so cut the side AB by the third demonstration, then we make ZL equal to AD and draw the arc LMN in such a way that the angle NLG is equal to BAG, LM is equal AH, ZM is equal to DH, LN is equal to AT, and GN is equal to ET, so MN comes to be equal to TH. MN is greater than BK, so TH is greater than BK. Since KM is greater than BN then the sum of BG and DH is less than the sum of ZK and TE.

The Eighth Demonstration: Also, if we make AD the same as GE and the rest stays as we mentioned, then I say that the arc GT is greater than the arc BH and AB is less than the sum of the two arcs DH and TE.

We make the angle ZDA equal to the angle BGA. The angle ZAD is equal to the angle TEG and the side AD is the same as EG, so the side AZ is equal to the side ET and the side DZ, which is greater than BH by the fourth demonstration, is equal to the side GT. Since the arc DH is greater than the arc BZ and the arc TE is equal to the arc AZ, then AB is less than the sum of DH and ET.

And in the same way, if we draw ET and DH to the side of AB and make the angles that AB faces equal to the angle BGA, which shows us in the way we mentioned that BG is less than the sum of the two arcs that come out as we mentioned.

Also, if the base is divided into two halves at the point D, and we draw DH in such a way that the angle GDH is equal to the angle BAG and draw DZ in such a way that the angle ZDA is equal to the angle BGA, then AB is less than twice DH and BG is less than twice DZ because GH is equal to DZ, DH is equal to AZ, DH is greater than BZ, and DZ is greater than BH.

The Ninth Demonstration: If the two legs of a triangle are not equal and its apex angle is not greater than a right angle, and its longest leg is not greater than a quarter-circle, and if we cut off from one of the legs two equal arcs and draw arcs from their endpoints to the base that make with it angles that are equal to the angle that the other leg encloses with the base, then these cut off from the base two unequal arcs and the greater of them is the one adjacent to the undivided leg. If the two equal arcs are cut off from the greater of the two legs then the sum of the least of the drawn arcs and the undivided side is less than the sum of the two remaining [drawn] arcs. If the two equal arcs are cut off from the lesser of the two legs then the sum of the least of the drawn arcs and the undivided leg is greater than the sum of the two remaining [drawn] arcs.

Let the triangle ABG be given and let the angle ABG be not greater than a right angle, and let each of the two arcs AB and BG be not greater than a quarter-circle. Let's cut off from one of the two arcs AB and BG two equal arcs, BD and EZ, and draw from the points D, E, and Z arcs to the base AG that enclose with it angles equal to the angle that is enclosed by the base AG and the side that is not divided, and faces the divided side; and let these arcs be DH, ET, and ZK. This is possible, by the first demonstration, because the two angles at the points A and G are less than two rights, and this is because the sum of the two sides AB, BG is less than a semicircle. Then I say that in the first picture the arc AH that is adjacent to the undivided side is greater than the arc TK.

We make the arc LH equal to the arc GK and construct over L the angle ALM, making it equal to the angle AGB. Then the arc LM cuts the side AB, by the third demonstration, because the arc AB is not greater than a quarter-circle and the sum of the two sides AB, BG is less than a semicircle. In the four-sided figure BDNM the side MN is greater than the side BD and the side DN is greater than the side BM by the fourth figure, so we make the arc NS equal to the arc BD and draw the arc SO in such a way that the angle SOG equals the angle BAG. The arc LH equals the arc GK, so the arc GZ is equal to the arc LN, by the fourteenth demonstration of the first treatise. The arc NS equals the arc ZE and thus is equal to BD, so the total arc SL is equal to the total arc GE. So in the two triangles SOL and ETG, the two angles on LO are equal to the two corresponding angles on GT, the side SL subtending one of these angles in one triangle is equal to the side EG subtending the corresponding angle in the other triangle, and each of the two remaining sides, SO and ET, is less than a quarter-circle and therefore their sum is less than a semicircle; so the arc OL is equal to TG by the seventeenth demonstration of the first treatise. Since the arc LH is equal to the arc GK then it follows that the arc OH is equal to the arc TK. In the second picture we see that from what we mentioned.

Also, if the two [equal] arcs are consecutive, such as BD and DE, and we follow the same path of this proof, then it becomes clear that of these arcs that were cut off as we described, the one adjacent to the undivided side is greater than the other segment.

The Tenth Demonstration: Afterwards, let BG be greater than AB, and let's cut off from BG two equal arcs and call them BD and EZ, and do the same as we did earlier. Then I say that the sum of the two arcs AB and ZK is less than the sum of the two arcs DH and ET.

We first make the angle at A right. Since we already showed that the arc AH is greater than the arc TK then we cut off from the arc AH the arc AL, equal to the arc TK, and we extend the arc BA to M, making AM the same as ZK, and we draw the two arcs LM and TZ. Then the two arcs AL and AM are equal to KT and KZ, respectively, and these pairs of sides enclose equal angles. So the arc LM is equal to the arc TZ, and the angle AML is equal to the angle KZT. Since the sum of the angles TKZ, TZK, and KTZ is not less than two right angles, the angle TKZ is equal to the angle THE, and the sum of the angles THE, ETZ, and ZTK is equal to two right angles, then the angle KZT is greater than the angle ETZ. So the angle AML is greater than the angle ETZ. We make the arc TN the same as the arc BM, and draw the arcs BL, BH, NZ. Since the two sides BM and ML are equal to the two sides NT and TZ, respectively, and the angle AML is greater than the angle NTZ, then the arc BL is greater than the arc NZ. The arc BH is greater than the arc BL, so the arc BH is greater than the arc NZ by a lot. Let the angle ZES be equal to the angle BDH. Then the arc BH is greater than the arc ZS, which is greater than the arc EZ, which is equal to the arc BD. So we extend TZ to O, making ZO equal to BH. In the two triangles BHD and EZO, since the angle ZEO from one of them is equal to the angle BDH from the other, and the sides that enclose the angles at B and Z are equal to their correspondents, and the sum of the two remaining angles BHD and ZOE is not equal to two right angles because they are both acute, then the side DH is equal to the side EO by the seventeenth demonstration of the first treatise. EO is greater than EN because ZO, being the same as BH, is greater than ZN, so DH is greater than EN. So then sum of the two arcs DH and ET is greater than the arc TN, which is the same as BM, which is the same as the sum of the two arcs AB and KZ.

As for him saying that the angle KZT would be greater than the angle ETZ, since the angles ZKG and ETK are equal then if we draw the two arcs TE and KZ until they meet, the sum of the arc between E and the point where they meet and the arc between Z and this point is less than a semicircle. So the angle KZT is greater than the alternate angle ETZ.

And as for him saying that BH would be greater than BD, since the angle DHG is supposed to be right—and this would be the same if it is obtuse—then the angle DHB is acute, and since BG is not greater than a quarter-circle then GD is less than a quarter-circle. The angle G is acute, so DH, which determines the angle at D that is subtended by DG if it is right or obtuse, is less than DG and therefore less than a quarter-circle.

Also, if the angle ABG is not greater than a right one then if AG is not greater than BG it is therefore not greater than a quarter-circle, so the two angles G and D are acute. So the angle BDH is obtuse and the angle DBH is acute, so BH is greater than BD. Since ZN is less than BL and BG is greater than BL then BH is much greater than ZN, as was mentioned.

And as for him saying "In the two triangles BHD and EZO", and what comes after that until he says "the side OE equals the side DH", this is as shown in the seventeenth demonstration of the first treatise.

And as for him saying that both of the two angles BHD and EOT are shown to be acute, the angle BHD is acute because the angle DHG is not smaller than a right angle. In the case of the angle EOT, ZO which equals BH is greater than TE because BH is greater than AB, which is greater than TE. So TO is much greater than ET, and ET is less than a quarter-circle. If the angle O is right or obtuse then TO is less than TE, since TE is less than a quarter-circle, so the angle EOT is acute by the seventeenth demonstration of the first treatise, and so EO is equal to DH.

And as for him saying that since ZO is greater than ZN then EO is greater than NE, if we connect NO by an arc from a great circle then the angle ONZ is greater than the angle NOZ because ZO is greater than ZN, and therefore the angle ONE is greater than the angle NOZ. So the angle ONE is much greater than the angle NOE, so the side EO is greater than EN. We already see that if the angle A is right or obtuse then the proof is the same.

The Eleventh Demonstration: Also, if we make the angle A acute and draw the arc BL, making it the same as BA, and make the arc DM equal to the arc DH, the arc EN equal to the arc ET, and the arc ZS equal to the arc ZK: then the arcs BL, DM, EN, ZS make with the base AG angles that are equal on the side of A. Therefore the angles they make on the side of G are also equal, and these are obtuse and the angles on the side of A are acute. For the sake of what has been presented, the sum of the two arcs DM and EN is greater than the sum of the two arcs BL and ZS, so the sum of the two arcs AB and ZK is less than the sum of the two arcs DH and ET because AB is equal to BL, ZK is equal to ZS, DM is equal to DH, and EN is equal to ET.

The Twelfth Demonstration: And we should imagine in this proof, all the cases of this demonstration, that we make the angle that is at A acute, and also make the arc BD equal to the arc EG where their sum is less than BG or greater than it. Then I say that the arc AH is greater than the arc GT.

We make the angle AHN equal to the angle AGB. Then NH is greater than DB by the fourth demonstration of this treatise, so we make HS the same as BD and do as we did before. Then since HS is the same as BD, which is the same as GE, the angles on the bases OH and TG are equal to their correspondents, and each of the arcs SO and ET is less than a quarter-circle and their sum is therefore less than a semicircle, then by the seventeenth demonstration of the first treatise the arc OH is equal to the arc TG. So the arc AH is greater than the arc TG. Also, if we divide the side BG into two halves at D and illustrate what we presented, then NH is greater than DB by the fourth demonstration of this treatise, and if we make HS equal to BD and draw SO, we again see by the seventeenth demonstration of the first treatise that OH is equal to HG.

The Thirteenth Demonstration: And I say also that the arc AB is less than the sum of the two arcs DH and ET.

We extend the arc TE to make TK equal to AB, and we make AZ equal to GT and draw the two arcs BZ and GK. Then the two sides BA and AZ are equal to the two sides KT and TG, respectively, and these pairs of sides enclose equal angles, so the arc BZ is equal to the arc KG. The arc BH is greater than the arc BZ and therefore greater than the arc KG.

Also, if we don’t make the angle at A greater than a right angle and we make the angle GEL equal to the angle BDH, then BH is greater than GL. BH is also greater than GE because it is greater than BD. We draw the arc GM, making it equal to the arc BH, and draw the arc EM. Since the side BG is not greater than a quarter-circle then BD is less than a quarter-circle by a lot, so the sum of the two arcs BD and BH is less than a semicircle, so the angle BDH is obtuse because it is greater than completing two right angles with the angle B, which is not greater than a right angle. Each of DH and BH is less than a quarter-circle, so the angle BHD is acute.

Also, because the sum of the two sides DB and BH is less than a semicircle, the sum of the two sides EG and GM is also less than a semicircle. The angle MEG is obtuse, so the angle EMG is acute. In the triangles BHD and GME, the angle GEM is equal to the angle BDH and the two sides that enclose the angle HBD are equal to their correspondents enclosing the angle MGE, and the sum of the remaining two angles at H and M is not equal to two right angles, so the side EM is equal to the side DH. EM is greater than EK, so DH is greater than EK. So the sum of DH and ET is greater than TK and therefore greater than AB.

This proof is similar to what is presented, and what he acknowledges in it is built on what we pointed to in the previous demonstrations. As for the angle MEG, where it is made to equal the obtuse angle BDH, and if MG becomes equal to BH where EG is already equal to BD, then knowing that these two together are less than a semicircle shows us that the angle at M is acute. Similarly, in the triangle HBD we show that the angle at H is acute. But we do not need that if we assume that the angle ABG is not greater than a right angle where we need to show that the angle at M is acute. If it is a right angle and the angle at H in the triangle BDH is acute, the proof would not be accurate. But observe that the sum of the two angles at M and H is not equal to two right angles, and so if we do that in the remaining two pictures then we see there what we want to show.

As for the divided one being the smaller side, it is only stated since the proof of that in this way cannot be accurate. So if it is shown here that KG is smaller than BZ and that in the earlier figure NZ is smaller than BL, then BH there and here is going to be greater than the arc that each one of the two arcs mentioned is less than. But if AB is the greater side and in the earlier figure NZ is smaller than BL, then for BH to be smaller than BL, NZ would be greater than BH so if it was given like this then the proof would be the same. On the other hand, if one makes one angle that is on the side to be equal to the angle BDH, he would make it smaller than the obtuse angle that we get from extending TE, and for that one would need to say in the proof that since the angles at T and E are equal then extending TE and HD until they meet would make the sum of these arcs equal to a semicircle. So the sum of the two sides that are from the two points D, E to the point of intersection is less than a semicircle. So the angle that is at E, which is the exterior of this triangle whose legs add up to less than a semicircle, is greater than the interior angle facing towards it at the point D which is the vertical angle to the angle BDH.

As for the proof that the divided one is the smaller side and the rest stays as it is so that the sum of the greater side and the smaller arc is greater than the sum of the two remaining arcs, it is then based on what Menelaus mentioned in the thirteenth demonstration of this treatise which is what was presented. So now let us repeat the triangle ABG where we let BG be the greater side and the rest be the same. We make BD and EZ from the lesser leg AB equal, and we take out the arcs DH, ET, ZK in such a way that the angle DHA, ETA, ZKA are equal to the angle

BGA. Then we say that the sum of BG and ZK is greater than the sum of DH and ET.

We make the arc GN equal to the arc ZK, the arc GM equal to the arc ET, and the arc LG equal to the arc DH. We draw the arcs NF, MO, and LS in such a way that the angle NFG, MOG, and LSG are equal to the angle BAG. Since NG is equal to ZK, the two angles F and A are equal, and the two angles G and K are also equal, then NF is equal to AZ. In the same way, we show that MO is equal to AE and LS is equal to DA. Since BD is equal to EZ then the sum of BA and AZ is equal to the sum of DA and AE. Since AZ is equal to FN, AE is equal to OM, and AD is equal to SL, then the sum of AB and NF is equal to the sum of the two arcs LS and MO. So by the fourteenth demonstration, BL is greater than MN so the sum of BG and GN is greater than the sum of LG and GM. Since GN is equal to ZK, GM is equal to ET, and LG is equal to DH, then the sum of the two arcs BG and ZK is greater than the sum of the two arcs ET and DH, and that is what Menelaus says.

Afterwards, let this be this way and the sum of BG and ZK be equal to the sum of ET and DH. Then I say that BD is smaller than EZ. We also make NG, MG, and LG equal to the arcs ZK, ET, and DH, respectively, and we draw the arcs LS, MO, NF so that the angles that are on the side of G are equal to the angle BAG. If the sum of BG and NG is equal to the sum of LG and GM because GN equals ZK, GM equals ET, and LG equals DH. Since the sum of BH and BG equals the sum of LG and GM then BL is equal to MN. So the sum of AB and NF is smaller than the sum of the two arcs LS and MO. Since NG equals ZK, the two angles of F and A that are on the side of G are equal by what is presented in the eleventh and twelfth demonstrations of this treatise, and the two angles of K and G that are on the side of A are equal: then AZ is equal to NF. In the same way, AE is equal to MO and AD is equal to LS. So the sum of AB and AZ is less than the sum of DA and AE. So BD is smaller than EZ.

The Fourteenth Demonstration: If two legs of a triangle are not equal and the angle at its apex is not greater than a right angle, and the greater of the legs is not greater than a quarter-circle, and if we draw between one of the legs and the base three arcs that cut off two equal arcs from the base and two equal arcs from the leg, where two of the arcs make angles with the base such that each is equal to the angle enclosed by the undivided leg and the base, then the angle made by the remaining arc is not equal to the angle that we mentioned. If the remaining arc is less than the first two drawn arcs then it makes a greater angle, and if not then it makes a lesser angle.

Let the triangle ABG be given and let the side BG be greater than the side AB—and thus the angle at B is not greater than a right angle—and let BG be not greater than a quarter-circle. Let's draw the arcs DH, ET, ZK, and let the arc BD be equal to the arc EZ and the arc AH be equal to the arc TK, and also let each of the two angles DHG and ETG be equal to the angle BAG. Then I say that the angle ZKG is greater than the angle BAG.

We make the arc LH equal to the arc KG and make the angle ALN equal to the angle BGA. Then the arc LN is equal to the arc GE, and since the arc NM is greater than the arc BD, which is equal to EZ, then we make the arc NS equal to the arc EZ and draw the arc SH. The arc SL is then equal to the arc GZ, and the arc LH is already equal to the arc GK and these pairs of sides enclose equal angles, so the angle SHL, which is greater than the angle BAG, is equal to the angle ZKG.

The Fifteenth Demonstration: Also, if we make the angle ZKG equal to each of the two angles DHG and BAG, then I say that the angle ETG is less than the angle BAG.

We make the arc LH equal to the arc KG and make the angle ALN equal to the angle AGB. Then the arc LM is equal to the arc ZG. Since the arc MN is greater than the arc BD then it is greater than EZ, so we make the arc MS equal to the arc EZ and draw the arc AS. Since LS equals GE and the angle SLA equals the angle AGB and the side GT equals the side LA, then ET is equal to SA and the angle ETG is equal to the angle SAL.

Let whoever looks in this book remember that from the fourth demonstration of this treatise, Menelaus has the conditions that the angle B is not greater than a right one and neither of the legs is greater than a quarter-circle, so that the arc MN in this demonstration, and its correspondent in each demonstration that is similar in this meaning, is greater than the arc BD that faces it in the four-sided figure by that proof in the fourth demonstration of this treatise.

The Sixteenth Demonstration: If each of the two angles on the base of a triangle is acute and neither of its legs is bigger than a quarter-circle, and if we cut off from the leg that is not greater than its companion two equal arcs and draw from their endpoints to the base arcs that enclose with it angles that are equal to the angle enclosed by the base and the undivided side, then these arcs cut off from the base two arcs that are not equal. The greater of the two is the one adjacent to the leg that is not divided.

Let the triangle ABG be given and let each one of its two angles at the points A and G be acute, and let each of the two legs AB and BG be not greater than a quarter-circle, and let BG be not greater than AB. Let's cut off from BG the two equal arcs BD and EZ, and draw from the points D, E, Z to the base AG the arcs DH, ET, ZK such that the angles they make on the side of G are equal to the angle BAG. Then I say that AH is greater than TK.

First let the side BG be equal to the side AB. We draw the arcs BL, DM, EN, and ZS, making them perpendicular to AG. Then the arc AG is twice the arc GL, and the arc GH twice the arc GM, and the arc GT twice the arc GN, and the arc GK twice the arc GS; so the arc AH is twice the arc LM and the arc KT is twice the arc NS. The arc BD is equal to the arc EZ, so the arc LM is greater than the arc NS by the ninth demonstration, and therefore the arc AH is greater than the arc TK.

The Seventeenth Demonstration: And now let BG be less than AB, and we do as we did earlier. Then we draw the arc BO to be equal to AB, the arc DF to be equal to the arc DH, the arc EQ to be equal to the arc ET, and the arc ZR to be equal to the arc ZK. Then the arc AO is twice the arc OL, and the arc HF twice the arc FM, and the arc TQ twice the arc QN, and the arc RK twice the arc RS; so it follows that the difference between the two arcs AO and HF is twice the sum of LM and OF, and the difference between the two arcs TQ and KR is twice the sum of NS and QR. The arc LM is greater than the arc NS because BD is equal to EZ, by the ninth demonstration of this treatise, and in the same way OF is greater than QR, so the remaining arc AH is greater than the remaining arc KT.

The Eighteenth Demonstration: If there is a great circle in a sphere and out of some parallel circles one is tangent to it, and if we cut off from the great circle two equal arcs between the tangency point and the greatest of these parallel circles, drawing parallel circles through the endpoints of these arcs and the pole of the parallel circles: then the parallel circles cut off unequal arcs from the circles coming out from the pole. The closer these arcs are to the greatest of the parallel circles, the greater they are. Also, of the arcs that get cut off from the greatest of the parallel circles, the closer they are to intersecting the great circle inclined towards the diameter, the less they are.

Let a great circle on a sphere be TMB, and tangent to it one of the parallel circles, which has DE on it, and let the greatest of the parallel circles be QRXVB. From TB we cut off two equal arcs, TK and LM, and draw through the pole H and the points T, K, L, M the great circles HTQ, HKR, HLX, HMV, and we draw through the points K, L, and M the parallel circles KS, LO, and MF. Then I say that the arc FO is greater than the arc ST and that the arc QR is greater than the arc XV.

The figure MHT is a triangle, and the side MH is greater than the side HT and each of the two is less than a quarter-circle. We already cut off from TM the two equal arcs TK and LM, so by the thirty-seventh demonstration of the first treatise the angle THK must be greater than the angle LHM. So the arc QR is greater than the arc XV. As shown in the demonstration we mentioned, the sum of the two arcs MH and HT is greater than the sum of the two arcs LH and HK. The point H is a pole of the parallel circles, so therefore the sum of the two arcs HF and HT is greater than the sum of the two arcs OH and HS, and one can see from this that OF is greater than ST.

And the benefit of this figure is that if TB is the ecliptic and BQ is the equator and equal arcs are cut off starting from the tangency point, i.e. from the solstice, then the ascension of the one closer to the solstice in the upright sphere is greater than the ascension of the farther one. The argument [of inclination] of the arc that is closer to the solstice is less than the argument of the arc that is closer to the point of equinox. R and Q are the ascensions of K and T in the upright sphere, and X and V are the ascensions of L and M in the upright sphere. Since the point T is the tangency point then QT is the greatest inclination. FQ is the inclination of BM because MF is one of the parallel circles. Since OQ is the inclination of BL then OF is the argument of inclination of LM. Since QS is the inclination of BK then ST is the argument of inclination of KT.

The Nineteenth Demonstration: If two great circles intersect and we cut off from one of them two equal arcs that have equal distance from a point of intersection of the two circles, and we draw circles through the endpoints of the two arcs and through a pole of one of the two circles, then these circles cut off from the other great circle two equal arcs.

Let two of the great circles on a sphere be ABE and HTL and let them intersect at the point G, and let's cut off from one of the two circles the two arcs AB and DE and let their distance from the point G be equal. Let's draw through the points A, B, D, E, and through a pole of one of the two great circles, the great-circle arcs ZAH, ZBT, ZKD, and ZLE. Then I say that the arc HT is equal to the arc KL.

We first make the point Z a pole of the circle ABDE. Then each of the two angles HAG and LEG is a right angle. The angle LGE is equal to the angle HGA and the arc GA is equal to the arc GE, so the side GH is equal to the side GL. In the same way we also show that the arc TG is equal to the arc KG, so it follows that the arc TH is equal to the arc KL.

Also, if we now make Z a pole of the circle LGH then each of the two angles AHG and ELG is a right angle. The angle AGH is equal to the angle LGE, the arc AG is equal to the arc EG, and the sum of the two arcs AH and LE is not equal to a semicircle because each of them is less than a quarter-circle, so the arc HG is equal to the arc LG. In the same way we show that the arc TG is equal to the arc KG, so it remains that the arc TH is equal to the arc KL.

From what he mentions in the proof when he makes the point Z a pole to HGL, where AG and GE are already equal in the circle ABDE and such that the sum of AH and LE is not equal to a semicircle because of the proof presented in the seventeenth demonstration of the first treatise and what is here: clearly we see that if equal arcs of the equator are of equal distance from the point of equinox then the parts of the ecliptic for one of them are equal to the parts of the ecliptic for the other one. Also, if equal arcs of the ecliptic are of equal distance from the point of equinox then the ascension of one in the upright sphere is equal to the ascension of the other in the upright sphere, and the inclinations of the equal arcs are equal for HA being equal to LE and TB being equal to KD. If we were to imagine that one of the two great circles is the horizon's circle and the other is the equator, then one can see from this demonstration that the Eastern amplitudes of parts at an equal distance from the point of intersection are the same. Also, their ascensions are the same. If TB is equal to KD then the two arcs that are of equal distance from the point of intersection coincide with the horizon HGL at the two points T and K, and GT equals GK. Also, GB is equal to GD, and they are the ascensions of the two parts arising from the horizon LGH at the two points T and K.

The Twentieth Demonstration: If there is a great circle in a sphere, and out of some parallel circles one is tangent to it, and if we cut off from the great circle two equal arcs between the tangency point and the greatest of these parallel circles, drawing parallel circles through the endpoints of these arcs, and drawing arcs of great circles through the endpoints of these arcs and either all through the two poles of the parallel circles or all tangent to a smaller one of the parallel circles than the first one we considered, in such a way that these great circles are inclined toward the same side as the first great circle we mentioned: then the parallel circles that have been drawn cut off from the great circles unequal arcs, the ones closer to the greatest parallel circle being greater than the ones farther; and of the arcs being cut off from the greatest parallel circle, the one closer to the intersection with the first great circle is less than the one farther.

Let the great circle AMB be given and the circle ADE, one of the parallel circles, be tangent to it, and let the greatest of the parallel circles be the circle GQZXVB. Let's cut off from the circle AMB the two equal arcs TK and LM, and draw through their endpoints the parallel circles KS, LO, MF and circles that either go through the two poles of the parallel circles or are all tangent to exactly one circle smaller than the circle ADE, inclined toward the side that AMB is already inclined toward. Then I say that FO is greater than TS and that QZ is greater than XV.

Since the angle TBG is less than a right angle and the angle BQT is not less than a right angle, then the side TQ is not greater than the side BT in the triangle BTQ. The side BT is not greater than a quarter-circle and the angle at the point T is not greater than a right angle, and we have already cut off the equal arcs TK and LM from the arc BT and drawn the arcs KZ, LX, MV that make with the base BG angles equal to the corresponding angle at Q, so the arc QZ is greater than the arc XV and the sum of the two arcs TQ and MV is less than the sum of the two arcs XL and KZ, by the nineteenth demonstration. Therefore the sum of the two arcs TQ and QF is less than the sum of the two arcs SQ and QO, so the arc FO is greater than the arc ST.

For the angle at the point T to not be greater than a right angle, we say that this happens because in the triangle BTQ, its angle at Q is not less than a right angle and each of the two sides enclosing the angle T is acute and therefore not greater than a right angle.

So it has been shown in this demonstration that if horizons have latitudes less than complementary to the inclination [of the ecliptic], then of equal arcs from the ecliptic, one closer to the solstice is one with greater ascension [over one of the horizons] and one closer to the point of equinox is one whose ascension over one of its horizons is less. As for the angles TQG, KZG, LXG, and MVG which the circles that are all tangent to a smaller one of the parallel circles make with the greatest of them, if we imagine AB as the ecliptic then the horizon whose latitude is supplemented by the quantity of the angle TQG translates to the horizon TQ with the point T from the ecliptic and with the point Q from the points M and V, whether the circle TQ is fixed or on parts of it [becomes] the circles going through the points T, K, L, and M, making the angles Q, Z, S, and V that are equal on one side. Then it cuts off these arcs from the equator by one ratio. Also, since OF is greater than ST then from this we also see that of equal arcs from the ecliptic, one that is closer to the point of the solstice is one whose argument of Eastern amplitude is greater. I mean that TQ is the Eastern amplitude of the point T, and FQ is the Eastern amplitude of the point M because it is equal to the arc MV. OF is the argument of Eastern amplitude of the arc LM because OQ is the Eastern amplitude of the point L. ST is the argument of Eastern amplitude of KT because SQ is the Eastern amplitude of the point K. Menelaus supposes that the inclination of the great circles that we drew is to the same side as the inclination of AMB against the parallel circles so that the angle T is not greater than a right angle, so his proof explains what is presented in the half of the ecliptic that is from the head of Capricorn to the head of Cancer.

As for the other half, if we repeat the circle AMB that is inclined over the parallel circles and the great circle GQZXVB as it is, and the inclination of the circles TQ, KZ, LX, MV is to the opposite side of the inclination of AMB in their construction over the parallel circles, and tangent to a circle as in the earlier case so that their inclination in both pictures is the same and the arc TK is equal to the arc LM: then the two angles at B and Q are acute. TW is tangent to a circle that is smaller than the circle ADE and therefore the point of tangency is inside the circle ADE, so neither of the two sides BT and TQ is greater than a quarter-circle. Already we've taken the two arcs TK and LM from the side BT to be equal, and we've drawn the arcs KZ, LX and MV that make equal angles at Z, X, and V on the side of B. TQ is smaller than BT, so if the angle at T is not obtuse then QZ is greater than VX.

If the angle is obtuse, then this is the same as shown in the seventeenth demonstration, or even in the sixteenth if BT is equal to TQ. But the angle of T is obtuse and BT is greater than TQ is supposed to be from Cancer's head to Capricorn's head.

The Twenty-First Demonstration: If a great circle in a sphere is tangent to one of some parallel circles, and if we cut off from it two equal arcs between the tangency point and the greatest of the parallel circles, drawing parallel circles through the endpoints of these arcs and drawing arcs of great circles through the endpoints of these arcs that are all tangent to a larger one of the parallel circles than the first one—and it's not necessary that their inclination be to the same side as the first of the great circles—then the parallel circles cut off from the great circle arcs that are not equal, and the smaller of them is the one closer to the great one of the parallel circles. The great circles that we drew cut off from the great one of the parallel circles arcs that are not equal, the one closer to the intersection with the first great circle being less than the farther one.

Let a great circle in a sphere be given with A and B on it, and tangent to it one of the parallel circles, which is ADE, and let the greatest of the parallel circles be BZQ. Let's cut off from the arc AB two equal arcs, TK and LM, and let's draw through their endpoints the parallel circles KS, LO, MF, and the great circles TQ, KZ, LX, MV, all tangent to exactly one of the parallel circles that are greater than circle ADE. Then I say that the arc FO is less than the arc ST and that the arc VX is less than the arc QZ.

The figure BQT is a triangle and the side QT is longer than the side TB, and the angle at T is not greater than a right angle, and we have already cut off from the side TB the two equal arcs TK and LM and drawn the arcs KZ, LX, and MV so that they make with the base angles equal to the corresponding angle at Q, so the arc QZ is greater than the arc XV.

Also, the sum of the two arcs TQ and MV is greater than the sum of the two arcs KZ and LX, and therefore the sum of the two arcs TQ and QF is greater than the sum of the two arcs SQ and QO. So it follows that the arc ST is greater than the arc OF. We know from what we said what has to be done in the reverse of all that.

What we should acknowledge about the angle T in what Menelaus presented in this book is that if the angle B is not greater than a right one then the angle T is obtuse if we assume that the angle Q is less than the angle B, because the sum of the three angles is greater than two right angles. Since BT is less than TQ and each of them is less than a quarter-circle then QZ is greater than XV by the seventeenth demonstration. Since the angle T is obtuse, then from him saying that the sum of TQ and MV is greater than the sum of KZ and XL, we see that if we extend QB from the point B [to C] making TC equal to TQ, and make KY equal to KZ and LW equal to LX and MR equal to MV: then we get the triangle BTC, and its apex angle is not greater than a right angle if the angle QTB is greater than a right angle. The divided side BT is less than TC, as Menelaus mentioned in the ninth demonstration, so the sum of TC and MR is greater than the sum of KY and LW, so the sum of TC and CN is greater than the sum of UC and CP. So TK, which equals ST, is greater than NP, which equals OF. CY is greater than RW as QZ is already greater than XV; for this reason Menelaus says that it is not necessary that the great circles whose inclinations are equal over the parallel circles be inclined to the same side as the first great circle.

5. Book III of Sphaerica

Premises by which knowing what is in the third treatise becomes easier:

1. If the two [straight] lines AB and AG meet at the point A and we draw from the two points B and G the two lines BE and GD, intersecting at the point Z, then I say that the ratio of GA to AE is the product of the ratio of GD to DZ and the ratio of ZB to BE.

We draw EH parallel to GD. Then the angle HEA is the same as the angle DGA and the angle A is shared by the two [planar] triangles AGD and AEH, so it follows that the third angles are the same and thus the two triangles are similar. So the ratio of GA to AE is the same as the ratio of GD to EH, which is the product of the ratio of GD to DZ and the ratio of DZ to EH. The ratio of DZ to EH is the same as the ratio of BZ to BE because of the similarity of the two triangles BHE and BDZ, so the ratio of GA to AE is the product of the ratio of GD to DZ and the ratio of BZ to BE.

2. Given the circle ABG with the center D, if the arc GA is less than a semicircle then I say that the ratio of the sine of the arc AB to the sine of the arc GB is the same as the ratio of AE to EG, where the chord AG is cut at E by the diameter coming out from B.

If the two arcs GB and BA are equal then the situation is clear, and if one of the two is less then let GB be the lesser one. Then we draw the perpendiculars GH and AZ onto the diameter, and these are the sines of the arcs GB and AB, respectively. Since the two triangles AEZ and GHE are similar then the ratio of GH to AZ is the same as the ratio of GE to EA.

3. Given the circle ABG with the center D, if we cut it by the two lines TDE and GBE which meet at the point E, then I say that the ratio of the sine of the arc GA to the sine of the arc BA is the same as the ratio of GE to BE.

We draw perpendiculars GH and BZ on TE. Since the angles at H and Z are right ones and the angle at E is shared by the two triangles HEG and ZEB, then the third angles are equal and so the two triangles are similar. So the ratio of GH to BZ is the same as the ratio of GE to EB.

4. If the ratio of A to B is the product of the ratio of G to D and the ratio of E to Z, then I say that the ratio of G the third to D the fourth is the

product of the ratio of A the first to B the second and the ratio of Z the sixth to E the fifth.

We make the ratio of H to T as the ratio of G to D and the ratio of T to J as the ratio of E to Z. Then since the ratio of H to J is the product of the ratio of H to T and the ratio of T to J, it is therefore the product of the ratio of G to D and the ratio of E to Z, so the ratio of H to J is as the ratio of A to B. The ratio of H to T, which equals the ratio of G to D, is the product of the ratio of H to J and the ratio of J to T, I mean the product of the ratio of A to B and the ratio of Z to E.

5. If the ratio of A to B is as the ratio of G to D, and the ratio of E to Z is the identity ratio, then I say that the ratio of A to B is the product of the ratio of G to D and the ratio of E to Z.

Let H be the same as B. Then the ratio of A to H is as the ratio of G to D and the ratio of H to B is as the ratio of E to Z. The ratio of A to B is the product of the ratio of A to H and the ratio of H to B, so it is the product of the ratio of G to D and the ratio of E to Z.*The First Demonstration:* Suppose the two arcs GE and BD meet at the point A and we draw from the two points G and B the two arcs GD and BE that intersect at the point Z, and each of these four arcs is from the circumference of a great circle in the sphere and each of them is less than half the circumference. Then I say that the ratio of the sine of arc GE to the sine of arc EA is the product of the ratio of the sine of arc GZ to the sine of arc ZD and the ratio of the sine of arc BD to the sine of arc BA.

Let H be the center of the sphere, and we draw the lines HZ, HB, and HE, and we join A and D [by a line]. Then the chord AD and the half-diameter BH are in one plane, so either HB is parallel to AD or it is not. If it is not, then they meet on one of the two sides, either on the side of D or on the side of A. If they meet on the side of D then let the meeting be at the point T, and we draw the chord AG which cuts the half-diameter EH, and let that be at the point K. We draw the chord GD which cuts the half-diameter ZH, and let that be at the point L. Since all the lines HE, HZ, and HT come out from the center of the circle of EZB to its circumference then all of them are in one plane, and the points K, L, and T are in this plane. The triangle AGD is in the plane of the two sides AG and AD, and the plane is taken out directly from this so that the point T is in this plane. So the points K, L, and T are in two planes, the plane of the circle EZB and the plane of the triangle AGD, so they are on the common section of the intersecting planes, which is a straight line. So the two lines GD and TK lie between the two lines AG and AT and intersect at L, so the ratio of GK to KA is the product of the ratio of GL to LD and the ratio of TD to TA. The ratio of GK to KA is as the ratio of the sine of arc GE to the sine of arc EA, the ratio of GL to LD is as the ratio of the sine of arc GZ to the sine of arc ZD, and the ratio of TD to TA is as the ratio of the sine of arc BD to the sine of arc BA, so the ratio of the sine of arc GE to the sine of arc EA is the product of the ratio of the sine of arc GZ to the sine of arc ZD and the ratio of the sine of arc BD to the sine of arc BA.

If HB and AD meet on the side of A then let the meeting be at the point T, and we draw the two arcs BDA and BZE until they meet at the diameter, and let this be at the point K. That shows that the ratio of the sine of GZ to the sine of ZD is the product of the ratio of the sine of GE to the sine of EA and the ratio of the sine of KA to the sine of KD. So the ratio of the sine of GE, the third, to the sine of EA, the fourth, is the product of the ratio of the sine GZ, the first, to the sine of ZD, the second, and the ratio of the sine of KD, the sixth, to the sine of KA, the fifth. Now, the sine of KD is the sine of BD and the sine of KA is the sine of BA so the ratio of the sine of GE to the sine of EA is the product of the sine of GZ to the sine of ZD and the ratio of the sine of BD to the sine of BA.

If AD is parallel to BH, then we complete the semicircle BAT and draw the two chords AG and DG, and we draw from D the perpendicular DS and from A the perpendicular AO [perpendicular to AD and to BH]. These are equal because the plane [that they enclose with AD and OS] is a parallelogram, so the sine of BD is the same as the sine of BA. We join from the center H the line EH, which cuts the chord AG at L. Since the diameter BT, the arc EZB, the line EH, and the point L are in one plane, then it is possible for us to draw from the point L and in the plane of EZH a line that is parallel to the diameter, and thus parallel to AD. It is also possible for us to draw a line parallel to AD from L and in the plane of ADG, and I say that [in both of these planes] it is the line LK.

Otherwise, let the parallel that comes out from L in the plane of EZB be the line LM and the one in the plane of ADG be the line LN. Then the two lines LM and LN are parallel and meet up at one point, which is a contradiction. So no line comes out from L parallel to AD except for the line LK. So in the triangle ADG a line parallel to the base has been drawn, so the ratio of GL to LA is as the ratio of GK to KD, so the ratio of the sine of GE to the sine of EA is as the ratio of the sine of GZ to the sine of ZD. The ratio of the sine of BD to the sine of BA is the identity ratio, so the ratio of the sine of GE to the sine of EA is the product of the ratio of the sine of GZ to the sine of ZD and the ratio of the sine of BD to the sine of BA.

I also say that the ratio of the sine of GA to the sine of AE is the product of the ratio of the sine of GD to the sine of DZ and the ratio of the sine of BZ to the sine of BE. We extend GA and GD so that they both meet on the [other] end of the diameter, and let this be T. The two arcs TE and BE have the two arcs TZ and BA intersecting between them, so the ratio of the sine of TA to the sine of AE is the product of the ratio of the sine of DT to the sine of DZ and the ratio of the sine of BZ to the sine of BE. Now, the sine of TA is the sine of GA and the sine of TD is the sine of GD, so the ratio of the sine of GA to the sine of AE is the product of the ratio of the sine of GD to the sine of DZ and the ratio of the sine of BZ to the sine of BE.

The Second Demonstration: If a pair of corresponding angles in two triangles are equal, and another pair of angles are either equal or add up to two right angles, then the ratio of the two sines of the two sides subtending the first pair of angles is as the ratio of the two sines of the other two sides subtending the other pair of angles.

Let the two triangles be ABG and DEZ and let the angle at A be equal to the angle at D, and let the angles at G and Z either be equal or add up to two right angles. Then I say that the ratio of the sine of AB to the sine of BG is as the ratio of the sine of DE to the sine of EZ.

We draw the two arcs GAH and BAT, making the arc AH equal to the arc DZ and the angle AHT equal to the angle EZD, and we complete the picture. Then the arc AT is equal to the arc DE and the arc TH is equal to the arc EZ. Since the two angles BGA and AHT are either equal or adding up to two right angles, then [GK and HK either add up to a semicircle or they are equal, and therefore] the sine of GK equals the sine of HK. Since the picture is the same [as the one for the first demonstration] then the ratio of the sine of GK to the sine of BG is the product of the ratio of the sine of KH to the sine of HT and the ratio of the sine of TA to the sine of AB. Since the sine of GK is equal to the sine of KH, then the ratio of the sine of HT to the sine of BG is as the ratio of the side of TA to the sine of BA. If we switch [the sine of TA with the sine of BG] then we preserve proportionality. The arc HT is equal to the arc EZ and the arc AT is equal to the arc DE, so the ratio of the sine of GB to the sine of AB is as the ratio of the sine of EZ to the sine of ED.

Also, if we make the angle at A equal to the angle at D and let the ratio of the sine of GB to the sine of AB be as the ratio of the sine of EZ to the sine of ED, then I say that either the two angles at the two points G and Z are equal or their sum is equal to two right angles.

If we do the same as the work mentioned, then the ratio of the sine of GB to the sine of AB is as the ratio of the sine of HT to the sine of TA. Again, if they were to be switched then we preserve proportionality, and since the picture is the same then the sine of KH is equal to the sine of KG. Therefore the two angles THA and AGB are either equal or add up to two right angles.

If one thinks through and measures between our work in the Enriched Demonstration and the work of Menelaus in the sector demonstration, and gathers what is needed from the proofs and has the knowledge that the two angles A and D are equal and the ratio of the sine of BG to the sine of BA is as the ratio of the sine of EZ to the sine of ED: then he can easily find, in the demonstration that is being improved which replaces the sector demonstration, that if the two sines of the two angles at Z and G are equal then either Z and G are equal or their sum is equal to two right angles.

The Third Demonstration: Given two triangles such that a base angle of each one is right and the remaining pair of base angles are equal but not right, then the ratio of the two sines of the two sides that enclose the right angle of one triangle is the product of the ratio of the two sines of the two sides that enclose the right angle of the other triangle--taken the same way that the first ratio is taken--and the ratio of the sine of the arc between the apex point of the first triangle and the pole of its base to the sine of the arc between the apex point of the other triangle and the pole of its base.

Let the two triangles be ABG and DEZ and let the two angles at A and D be right ones, and let the two angles at G and Z be equal but not right, and let the two poles of the two arcs AG and DZ be H and T, respectively.

Then I say that the ratio of the sine of AB to the sine of AG is the product of the ratio of the sine of ED to the sine of DZ and the ratio of the sine of BH to the sine of ET.

We make the arc GK equal to the arc DZ and draw the arc HK that cuts BG at the point L. Then the arc KL is equal to the arc DE and the arc LH is equal to the arc ET. Because of the picture being as it is, the ratio of the sine of AB to the sine of BH is the product of the ratio of the sine of AG to the sine of GK and the ratio of the sine of KL to the sine of LH, so the ratio of the sine of AB to the sine of AG is the product of the ratio of the sine of BH to the sine of LH and the ratio of the sine of LK to the sine of KG. The arc KG is equal to the arc DZ and the arc KL is equal to the arc DE and the arc LH is equal to the arc ET, so the ratio of the sine of AB to the sine of AG is the product of the ratio of the sine of DE to the sine of DZ and the ratio of the sine of BH to the sine of ET. It is also clear that the ratio of the sine of BG to the sine of GA is the product of the ratio of the sine of EZ to the sine of DZ and the ratio of the sine of BH to the sine of ET.

If it is as the book's author mentions, then showing by what is presented of the sector demonstration that the ratio of the sine of AB to the sine of BH is the product of the ratio of the sine of KL to the sine of LH and the ratio of the sine of AG to the sine of KG—and then saying that the ratio of the sine of AB to the sine of AG, the fifth, is the product of the ratio of the sine of BH, the second, to the sine of LH, the fourth, and the ratio of the sine of KL, the third, to the sine of KG, the sixth—needs additional proof. Since the ratio of the sine of AB to the sine of AG is as the ratio of the sine of the angle at G to the sine of the angle at B, and the ratio of the sine of ED to the sine of DZ is as the ratio of the sine of the angle at Z to the sine of the angle at E, and the two angles at Z and G are equal, then the ratio of the sine of AB to the sine of AG is the product of the ratio of the sine of ED to the sine of DZ and the ratio of the sine of the angle at E to the sine of the angle at B. But ratio of the sine of BH to the sine of LH is as the ratio of the sine of the angle at L, which is equal to the angle E, to the sine of the exterior angle at B whose sum with the interior angle is equal to two right angles. So the ratio of the sine of AB to the sine of AG is the product of the ratio of the sine of BH to the sine of LH, which is equal to the ratio of the sine of angle E to the sine of angle B, and the ratio of the sine of KL to the sine of KG.

The Fourth Demonstration: Given two triangles such that each angle on one base is equal to its correspondent and neither is right, and we draw a perpendicular from each triangle's apex point [to its base], then the sines of the arcs that get cut off from the base are proportional.

Let the two triangles be ABG and DEZ, and let the angle at A be equal to the angle at D and let the one at G be equal to the one at Z, and none of these right. Draw from the two points B and E two perpendiculars on the two bases AG and DZ, which are BH and ET. Then I say that the ratio of the sine of AH to the sine of HG is as the ratio of the sine of DT to the sine of TZ.

We make the two poles of the two arcs AG and DZ be the two points K and L. Since the angles at H and T are right angles, and the angles at D and A are equal, and the points K and L are the poles of the arcs AG and DZ, then the ratio of the sine of AH to the sine of DT is the product of the ratio of the sine of BH to the sine of ET and the ratio of the sine of EL to the sine of BK. Similarly, because the angles at G and Z are equal but not right then the ratio of the sine of GH to the sine of ZT is the product of the ratio of the sine of BH to the sine of ET and the ratio of the sine of EL to the sine of BK. Therefore the ratio of the sine of AH to the sine of DT is as the ratio of the sine of GH to the sine of ZT. If we switch [GH with DT] then we preserve proportionality.

And in regards to astronomy, this demonstration shows that the ratios of the sines of the risen arcs that are equal in their quantities and in the quantities of how far they would be from the point of equinox in the upright sphere, to the sines of their right ascensions in the total of the inclined sphere, are one ratio.

The Fifth Demonstration: Given two triangles such that one pair of corresponding base angles are acute and the other two are right, and each of the sides subtending the remaining pair of angles is less than a quarter-circle, then the ratio of the sine of the sum of the two arcs enclosing one of the equal acute angles to the sine of the difference between the two of them is as the ratio of the sine of the sum of the two arcs enclosing the other of the equal acute angles to the sine of the difference between them.

Let the two triangles be ABG and DEZ, and let the two angles at A and D be right angles and the two angles at G and Z be equal and acute, and let each of the two arcs GA and DZ be less than a quarter-circle.

Then I say that the ratio of the sine of the sum of the two arcs BG and GA to the sine of the difference between BG and GA is as the ratio of the sine of the sum of the two arcs EZ and ZD to the sine of the difference between EZ and ZD.

We extend the arc BG from G to L, making each of the two arcs GK and GL equal to the arc AG, and we draw the two arcs AK and AL. Making G a pole we draw a great-circle arc that has the points H, M, S, and N on it. The point H is also a pole to the arc AGS because each of the two arcs AH and HS is constructed to be perpendicular to AS, and we draw the two arcs GM and GN. Since the arc AG is equal to the arc GK and AG is not equal to a quarter-circle, and the arc GM is a quarter-circle, then the angle KGM is equal to the angle MGS. That also shows that the angle LGN is equal to the angle NGS.

We also extend the arc EZ from Z to F, making each of the two arcs ZF and ZO equal to the arc DZ, and we draw the two arcs DO and DF. Making Z a pole we draw an arc from a great circle that has the points T, Q, X, and R on it, and we see as we showed earlier that the line ZQ also divides the angle EZX into two halves and the line RZ divides the angle FZX into two halves, so the angle MGS is equal to the angle QZX and the angle NGS is equal to the angle RZX. The two points G and Z are poles of the two arcs HMSN and TQXR, respectively, so the arc MS is equal to the arc QX and the arc NS is equal to the arc RX, and therefore the arc MH is equal to the arc TQ.

Since the arcs AH, AM, AS, and AN have already come out from the point A to the two arcs BGL and HSN, then the ratio of the sine of LB to the sine of BK is the product of the ratio of the sine of LB to the sine of LG and the ratio of the sine of LG to the sine of GK and the ratio of the sine of GK to the sine of KB, because LG is equal to GK. This ratio is the same as the product of the ratio of the sine of NH to the sine of NS and the ratio of the sine of MS to the sine of MH. In the same way that also shows that the ratio of the sine of FE to the sine of EO is the product of the ratio of the sine of TR to the sine of RX and the ratio of the sine of QX to the sine of TQ. It has already been clear that the arcs HM, MS, and NS are equal to the arcs TQ, QX, and XR, respectively; so therefore the ratio of the sine of LB to the sine of KB is as the ratio of the sine of FE to the sine of EO.

Menelaus was being vague in proving this demonstration, and that was either because he loved for an observer to debate him in his book or because what he had in mind that was needed to finish the proof was the least he expected from an observer. We redraw the triangle ABG as it was, and we draw the two arcs AK, AL and extend KG to L as was done so that the three arcs AG, LG, and KG are equal.

And we say that the ratio that is the product of the ratio of the sine of LB to the sine of LG and the ratio of the sine of KG to the sine of KB—whether the arcs AG, LG, and KG are equal or not—is as the product of the ratio of the sine of angle BAL to the sine of angle GAL and the ratio of the sine of angle KAG to the sine of angle BAK, because the ratio of the sine of BL to the sine of AL is as the ratio of the sine of the angle BAL to the sine of the angle B and the ratio of the sine of AL to the sine of LG is as the ratio of the sine of the angle G to the sine of the angle GAL. So the ratio of the sine of BL to the sine of LG is as the ratio of the sine of angle BAL to the sine of the angle at B multiplied by the ratio of the sine of the angle G to the sine of angle GAL, and therefore it is as the product of the ratio of the sine of angle BAL the third to the sine of angle GAL the sixth and the ratio of the sine of angle G the fifth to the sine of angle B the fourth.

Also, the ratio of the sine of KG to the sine of AK is as the ratio of the sine of angle KAG to the sine of angle G, and the ratio of the sine of KA to the sine of BK is as the ratio of the sine of angle B to the sine of angle BAK, so the ratio of the sine of KG to the sine of BK is the product of the ratio of the sine of angle KAG to the sine of angle G and the ratio of the sine of angle B to the sine of angle BAK, so then it is as the product of the ratio of the sine of angle KAG the third to the sine of angle BAK the sixth and the ratio of sine angle B the fifth to the sine of angle G the fourth. Of these four ratios, the ratio of the sine of angle G to the sine of angle B is reciprocal to the ratio of the sine of angle B to the sine of angle G, so the product of the ratio of the sine of BL to the sine of LG and the ratio of the sine of KG to the sine of BK is as the product of the ratio of the sine of angle BAL to the sine of angle GAL and the ratio of the sine of angle KAG to the sine of angle BAK. This is how it is in all spherical triangles where two arcs come out to the base between two legs of a triangle. If KG is equal to LG in here then the product of the ratio of the sine of angle BAL to the sine of angle GAL and the ratio of the sine of angle KAG to the sine of angle KAB, if it is as we mentioned, is as the ratio of the sine of LB to the sine of KB.

We repeat the figure and draw the circles AM, AS, and AN until they meet at the point C, and we also draw the two circles HMSN and LGKB from the points H, N, L, and B until the two of them meet at the two points J and V. Since AG, GK, and LG are equal and G is a pole of HMS then the arcs SC, LJ, and KV are equal and the angles S, J, and V are right ones. In the two triangles SCN and LNJ the two corresponding angles are equal and the angle L evaluates the completion of the inclination of SN from the inclination of the one greater than it that evaluates the angle C, so the angles of the triangle SCN are equal to the [corresponding] angles of the triangle LNJ, so the sides of these two triangles are equal to their correspondents. The case is similar in the two triangles SCM and MVK, so the arc SN is equal to the arc NJ and the arc SM is equal to the arc MV.

We repeat the figure again, and making the point A a pole we draw by the distance of the side of the square the arc UW. It is clear that since the point H is a pole of circle AG then arc UW is from a circle that goes through the point H: let this be the circle HUWP. Then HU evaluates the angle BAK, UW evaluates the angle KAG, HP evaluates the angle BAL, and WP evaluates the angle GAL. So the ratio of the sine of LB to the sine of BK is as the ratio of the sine of HP to the sine of UP multiplied by the ratio of the sine of UW to the sine of HK. The ratio of the sine of HN to the sine of SN is the product of the ratio of the sine of HP to the sine of PW and the ratio of the sine of AW to the sine of AS, and the ratio of the sine of SM to the sine of MH is the product of the ratio of the sine of UW to the sine of UH and the ratio of the sine of AS to the sine of AW. Of these four ratios, the reciprocal of the ratio of the sine of AW to the sine of AS is the ratio of the sine of AS to the sine of AW, so it follows that the product of the ratio of the sine of HN to the sine of NS and the ratio of the sine of SM to the sine of MH is as the product of the ratio of the sine of HP to the sine of PW and the ratio of the sine of UW to the sine of HU. So the product of the ratio of the sine of HN to the sine of NS and the ratio of the sine of MS to the sine of MH is as the ratio of the sine of LB to the sine of BK, since the angle MGS is equal to the angle QZX and the angle XZR is equal to the angle SGN. HS is a quarter-circle as TX is a quarter-circle, and MN is a quarter-circle as QR is a quarter-circle, because GM divides the angle KGS into two halves and GN divides the angle SGL into two halves, as is the case in the second demonstration. Also, since MH and NS are equal and HS is a quarter-circle then the sine of HN equals the sine of MS, so the product of these two ratios is as the ratio of the sine of MS to the sine of MH multiplied by itself. Since the product of the ratio of the sine of HN to the sine of NS and the ratio of the sine of SM to the sine of MH is the product of the ratio of the sine of angle HAN to the sine of angle SAN and the ratio of the sine of angle SAM to the sine of angle HAM, as the product of the ratio of the sine of BL to the sine of LG and the ratio of the sine of KG to the sine of KB is also the product of those two ratios; and also, the product of the ratio of the sine of TR to the sine of RX and the ratio of the sine of QX to the sine of TQ is the product of the ratio of the sine of angle TDR to the sine of angle XDR and the ratio of the sine of angle QDX to the sine of angle TDQ, as the product of the ratio of the sine of EF to the sine of FZ and the ratio of the sine of OZ to the sine of OE is also the product of these two ratios; then when we showed that each of the two arcs QX and MS evaluates half of the angle KGS, each of the two arcs RX and NS evaluates half of angle SGL, and that TQ equals MH, it became clear to us from this that the ratio of the sine of LB to the sine of BK is as the ratio of the sine of FE to the sine of EO. From here it has been clear that the ratio of the sine of the sum of an arc of a star's orbit to its ascension in the upright sphere to the sine of their difference is as the ratio of the sine of half of the completion [to a semicircle] of the greatest inclination to the sine of half of the greatest inclination, multiplied by itself. That is to say, if the acute angle G evaluates the greatest inclination then its obtuse angle evaluates the completion of that [arc] to a semicircle; and if the arc GM divides the obtuse angle at G into two halves then MS is half of the completion of the greatest inclination to a semicircle; and HS is a quarter-circle, so MH is half of the greatest inclination.

The Sixth Demonstration: If an angle in a triangle is divided into two halves, then the two ratios of the sines of the sides to the sines of the divided parts of the base are two equal ratios. The reverse is also so, and also on switching.

Let the triangle ABG be given and let the arc BD divide the angle at B into two halves.

Then I say that the ratio of the sine of AB to the sine of AD is as the ratio of the sine of BG to the sine of DG.

The two figures ABD and GBD are triangles, and the angles ABD and GBD are equal and the sum of their angles at D is equal to two right angles, so the ratio of the sine of AB to the sine of AD is as the ratio of the sine of BG to the sine of GD, and if we switch then they are still proportional. Now let the ratio of the sine of AB to the sine of BG be as the ratio of the sine of AD to the sine of DG. Then I say that the arc BD divides the angle B into two halves.

Since the two angles at D are also equal to two right angles, and the ratio of the sine of AB to the sine of AD is as the ratio of the sine of BG to the sine of GD, and the sum of the angles ABD and DBG is not equal to two right angles, then the angle ABD is equal to the angle DBG. Menelaus built this over what he presented in the second demonstration, and by the Enriched Demonstration one can disregard this, so it is as the repeated one.

The Seventh Demonstration: Also, if we make the angle that is adjacent to the angle ABG divided into two halves by the arc BD, then I say that the ratio of the sine of AB to the sine of BG is as the ratio of the sine of AD to the sine of DG; and the reverse of that also.

The two figures ABD and GBD are triangles, and they share the angle at D and the sum of the two angles ABD and GBD is equal to two right angles, so the ratio of the sine of AB to the sine of AD is as the ratio of the sine of BG to the sine of GD. The switching of that also holds. The reverse is clear from the Enriched Demonstration.

The Eighth Demonstration: If two arcs come out of a triangle's apex point to its base that make with the two legs two equal angles, then the ratio that is composed of sines of divisions of the base is equal to the ratio of the two sines of the two legs to the [second] power. The reverse is true also.

Let the triangle ABG be given and let's draw the two arcs BD and BE from the point B to the base AG, and let the two angles ABD and GBE be equal. Then I say that the ratio of the square generated from the sine of AB to the square generated from the sine of BG is as the ratio of the rectangle enclosed by the two sines of EA and AD to the rectangle enclosed by the two sines of DG and GE.

From the point G to the two [extended] arcs BE and BD we draw the two arcs GZ and GH, in such a way that the angle GZB is equal to the angle ABE and the angle GHD is equal to the angle ABD. Then the ratio of the sine of AB to the sine of GZ is as the ratio of the sine of AE to the sine of EG, and [similarly] the ratio of the sine of AB to the sine of GH is as the ratio of the sine of AD to the sine of DG. So the ratio of the square generated from the sine of AB to the one [rectangle] that is from multiplying the sine of GH by the sine of GZ, is as the ratio of the one generated from multiplying the sine of AE by the sine of AD to the one that is from multiplying the sine of DG by the sine of EG. Since the angle BHG is equal to the angle GBZ and the angle BZG is equal to the angle GBH, then the one [rectangle] resulting from multiplying the sine of GH by the sine of GZ is equal to the square generated from the sine of BG. Therefore the ratio of the square generated from the sine of AB to the square generated from the sine of BG is as the ratio of the plane generated from the sine of AE by the sine of AD to the plane generated from the sine of GE by the sine of DG.

Know that this proof is correct only when BG is not a quarter-circle. If it is a quarter-circle then it is not possible to draw from the point G to the arc BD an arc enclosing with it an angle that is smaller than angle GBD. Also, if BG is a quarter-circle then there are no two sines such that the sine of BG is between them and is the mean in their ratios unless each of them is the whole sine.

So the general proof for whether BG is a quarter-circle, less, or greater, is the one we are going to mention. Since the angle GBE is equal to the angle ABD then the angle GBD is equal to the angle ABE. The ratio of the sine of AE to the sine of GD, if we make the two sines of AB and BG between the two of them two means in the ratio, is the product of the ratio of the sine of angle ABE to the sine of angle BEA and the ratio of the sine of AB to the sine of BG and the ratio of the sine of angle BDG to the sine of angle GBD. Now, the ratio that is the product of the ratio of the sine of angle BDG to the sine of angle GBD and the ratio of the sine of angle ABE to the sine of angle BEA is as the ratio of the sine of angle BDG to the sine of angle BEA because the two angles GBD and ABE are equal, so the ratio of the sine of AE to the sine of GD is as the ratio of the sine of angle BDG to the sine of angle BEA paired up with the ratio of the sine of AB to the sine of BG.

Also, with the ratio of the sine of AD to the sine of GE, we make the two sines of AB and BG two means between them in the ratio that is the product of the ratio of the sine of angle ABD to the sine of angle BDA and the ratio of the sine of AB to the sine of BG and the ratio of the sine of angle BEG to the sine of angle GBE. Angle GBE equals angle ABD, so the ratio that is the product of the ratio of the sine of angle BEG to the sine of angle GBE and the ratio of the sine of angle ABD to the sine of angle BDA is as [the product of] the ratio of the sine of angle E to the sine of angle D and the ratio of the sine of AB to the sine of BG. Of these four ratios, two of which compose the ratio of the sine of AE to the sine of GD and two of which compose the ratio of the sine of AD to the sine of GE, the ratio of the sine of angle E to the sine of angle D and the ratio of the sine of angle D to the sine of angle E are reciprocals, so it follows that the ratio that is the product of the ratio of the sine of AE to the sine of GD and the ratio of the sine of AD to the sine of GE is as the ratio of the sine of AB to the sine of BG, multiplied by itself.

The Ninth Demonstration: Also, if we make the ratio of the square generated from the sine of AB to the square generated from the sine of BG as the ratio of the rectangle that is enclosed by the two sines of the two arcs AE and AD to the rectangle that is enclosed by the two sines of the two arcs GD and GE, then I say that the angle ABD is equal to the angle GBE.

We extend the two arcs AB and GB to the two points Z and H, making the arc BZ equal to the arc AB and the arc BH equal to the arc GB, and we extend the arc DB to T. Then the arc ZT is equal to the arc AD. We draw from the point B an arc that encloses with BH an angle equal to the angle ZBT, which is the arc BK. Then the ratio of the square generated from the sine of BZ, which is the same as the sine of AB, to the square generated from the sine of BH, which is the same as the sine of GB, is as the ratio of the rectangle enclosed by the sines of the arcs KT and ZT to the rectangle enclosed by the sines of the arcs TH and KH. The arcs ZT and TH are equal to AD and DG, respectively, so this shows that the arc KH is also equal to the arc GE. The arcs BH and KH, which are equal to the two sides EG and BG, enclose an angle equal to the angle BGE, so the bases and other angles are equal. The angle KBH is already equal to the angle TBZ, which is equal to the angle ABD, so the angle ABD is equal to the angle GBE.

Also, one can show the reverse of what has been presented in this demonstration. Assume that the ratio of the sine of AE to the sine of GD is as we mentioned and the ratio of the sine of AD to the sine of GE is as the one we showed. Then if the ratio of the square of the sine of AB to the square of the sine of BG is as the ratio of the rectangle enclosed by the sines of AE and AD to the rectangle enclosed by the the sines of GD and GE, we can see by what we presented that the ratio of the sine of angle E to the sine of angle GBE, if it is duplicated by the ratio of the sine of angle ABD to the sine of angle D, is as the ratio of the sine of angle E to the sine of angle D. So it is clear that the sine of angle GBE is equal to the sine of angle ABD; and the sum of these two angles is not equal to two right angles, so these two angles are equal.

The Tenth Demonstration: If there is a triangle with a right angle, and two arcs come out from the right angle to [an extension of] the base so that they enclose with one of the legs two angles that are equal, then the ratio of the sine of the sum of the base and the arc that is added to it, to the sine of the added arc, is as the ratio of the sine of the part—

out of the two parts of the base—that is not adjacent to the leg that we mentioned, to the sine of the other part. The reverse is also true.

Let the triangle ABG be given and let its angle at the point B be right, and let the two arcs coming out from B to [an extension of] the base be BD and BE, and let them enclose with the arc AB two angles that are equal. Then I say that the ratio of the sine of GE to the sine of AE is as the ratio of the sine of GD to the sine of DA.

Since the angle ABG is right and the angle ABD is equal to the angle ABE, then the arc BG divides the angle that is adjacent to the angle EBD into two halves. So the ratio of the sine of BE to the sine of BD is as the ratio of the sine of EG to the sine of GD, and is as the ratio of the sine of EA to the sine of AD, and therefore the ratio of the sine of EG to the sine of GD is as the ratio of the sine of EA to the sine of AD. If we switch then the proportion still holds, so the ratio of the sine of GE to the sine of EA is as the ratio of GD to the sine of AD.

Also, if we make the ratio of the sine of GE to the sine of EA as the ratio of the sine of GD to the sine of DA and the angle EBA equal to the angle ABD, then I say that the angle ABG is right.

If we switch, the ratio of the sine of EG to the sine of GD is as the ratio of the sine of EA to the sine of AD. The ratio of the sine of AE to the sine of AD is as the ratio of the sine of EB to the sine of BD, so the arc BG divides the angle that is adjacent to angle EBD into two halves, and from that the angle ABG must be right.

Also, since the angle ABE is equal to the angle ABD and the angle ABG is right, then the sine of the angle GBE is equal to the sine of the angle GBD. The ratio of the sine of GE to the sine of EA is as the ratio of the sine of angle GBE to the sine of angle ABE multiplied by the ratio of the sine of angle A to the sine of angle G. Also, the ratio of the sine of GD to the sine of DA is as the ratio of the sine of angle GBD to the sine of angle DBA multiplied by the ratio of the sine of angle A to the sine of angle G. So the ratio of the sine of GE to the sine of AE is as the ratio of the sine of GD to the sine of DA.

The Eleventh Demonstration: Also, if we now make the ratio of the sine of GE to the sine of EA as the ratio of the sine of GD to the sine of AD, and make the angle ABG right, then I say that the angle EBA is equal to the angle ABD.

We extend the two arcs AB and GB, making the arc BZ equal to the arc AB and the arc BH equal to the arc BG, and we draw the two arcs EBT and HZT so that the two arcs HZ and ZT are equal to the arcs GA and AE, respectively. We make the angle KBZ equal to the angle TBZ. The angle HBZ is right, so the ratio of the sine of TH to the sine of TZ is as the ratio of the sine of HK to the sine of KZ. The ratio of the sine of TH to the sine of TZ is as the ratio of the sine of GE to the sine of EA, because each of the two arcs is equal to one of the two arcs that we mentioned before, and this ratio is as the sine of GD to the sine of DA. The arc ZH is equal to the arc GA, so the arc ZK is equal to the arc AD. The arc ZB is equal to the arc BA, and these pairs of arcs enclose equal angles, so the angle KBZ is equal to the angle ABD. The angle KBZ is equal to the angle ZBT, which is equal to the angle ABE, so the angle ABD is equal to the angle ABE.

In the reverse of what we mentioned in the proof, if the ratio of the sine of GE to the sine of AE is as the ratio of the sine of GD to the sine of DA, and the angle ABG is right, then the ratio of the sine of GE to the sine of AE is as the ratio of the sine of angle A to the sine of angle G multiplied by the ratio of the sine of angle GBE to the sine of angle ABE. The ratio of the sine of GD to the sine of AD is as the ratio of the sine of angle A also to the sine of angle G multiplied by the ratio of the sine of angle GBD to the sine of angle DBA. The angle ABG is right, so the angle ABE, which is the complement of the angle GBD in the right angle, is equal to the angle DBA, which is how much less than a right angle the angle GBD is.

The Twelfth Demonstration: If any two of the angles of a triangle are each divided into two halves, and if we draw an arc from the place where the arcs dividing these two angles meet to the third angle, then it divides the third angle into two halves.

Let the triangle ABG be given and let the two angles at the points A and G be bisected by the two arcs AD and GD, respectively, and let the two points D and B be connected by the arc DB.

Then I say that DB divides the angle ABG into two halves.

We extend the arc BD to meet the arc AG at the point E. Since the two angles that are at the two points A and G are divided by the two arcs AD and GD, respectively, into two halves, then the ratio of the sine of BD to the sine of DE is as the ratio of the sine of BG to the sine of GE, and as the ratio of the sine of BA to the sine of AE, by the sixth demonstration. If we switch then the ratio of the sine of BG to the sine of BA is as the ratio of the sine of GE to the sine of AE, so the angle ABG is divided by the arc BD into two halves, by the reverse of the sixth demonstration.

A different way of this is that, since the ratio of the sine of GD to the sine of BD is as the ratio of the sine of angle DBG to the sine of angle BGD and the ratio of the sine of BD to the sine of AD is as the ratio of the sine of angle BAD to the sine of angle ABD, then the ratio of the sine of GD to the sine of AD is the product of the ratio of the sine of the angle GBD the third to the sine of angle DBA the sixth and the ratio of the sine of angle DAB the fifth to the sine of angle BGD the fourth. The ratio of the sine of GD to the sine of AD is as the ratio of the sine of angle BAD to the sine of angle BGD because the two arcs GD and AD divide the two angles at G and A into two halves, so the sine of angle GBD equals the sine of angle DBA. These two angles are not equal to two right angles, so they are equal.

The Thirteenth Demonstration: If we draw two arcs of the angles of a triangle, perpendicular to the two sides that are facing these two angles, then the arc that comes out from the remaining angle to the place where the first two arcs meet is perpendicular to the remaining side.

Let the triangle ABG be given, and let the perpendiculars AD and GE come out from the two points A and G to the two sides BG and BA and meet at the point Z, and let’s draw the arc ZB and extend it to the arc AG, to meet it at the point H.

Then I say that the arc BH is perpendicular to the side AG.

We draw the arcs EH, DH, and ED, extend ED until it meets the arc AG at the point T. Since the picture is as it is then the ratio of the sine of AT to the sine of TG is as the ratio of the sine of AH to the sine of HG, because each one of these is as the product of the ratio of the sine of AD to the sine of ZD and the ratio of the sine of ZE to the sine of EG and that is because the ratio of the sine of AH to the sine of HG is as the product of the ratio of the sine of AZ to the sine of ZD and the ratio of the sine of EZ to the sine of EG and the ratio of the sine of AD to the sine of AZ. Each of the two angles AEG and ADG is right so the angle DEG is equal to the angle GEH, and in the same way the angle ADH is equal to the angle ADE. Since the figure DEH is a triangle and its two angles at the points D and E are divided by the two arcs EZ and ZD into two halves, and the arc ZH is drawn, then by the previous demonstration the angle DHE is also divided into two halves by the arc HK. Since the picture is as it is then the ratio of the sine of ET to the sine of TD is as the ratio of the sine of EK to the sine of KD, because each one of them is the product of the ratio of the sine of EG to the sine of GZ and the ratio of the sine of ZA to the sine of AD. So the angle BHG is a right angle.

As for him saying that the ratio of the sine of AT to the sine of TG is as the ratio of the sine of AH to the sine of HG: since the ratio of the sine of AH to the sine of HG is as the product of the ratio of the sine of AZ to the sine of ZD and the ratio of the sine of BD to the sine of BG, but the ratio of the sine of BD to the sine of BG is the product of the ratio of the sine of EZ to the sine of EG and the ratio of the sine of AD to the sine of AZ, therefore the ratio of the sine of AH to the sine of HG is the product of the ratio of the sine of AD to the sine of DZ and the ratio of the sine of EZ to the sine of EG. The ratio of the sine of AT to the sine of TG is also the product of these two ratios. Since this is like this and the angle ADG is right then the angle TDG is equal to the angle GDH, by the eleventh demonstration, and since the two angles GDT, GDE are equal to two right angles and the angle ADG is right then the sum of the two angles GDT and ADE is right. The angle TDG is equal to the angle GDH, so the angle ADH is equal to the angle ADE. We extend GA from A and TE from E until they meet at S. We can see that the ratio of the sine of TG to the sine of TA is as the ratio of the sine of GH to the sine of HA, but the sine of TG is the sine of SG and the sine of TA is the sine of SA because TAS is a semicircle, so the ratio of the sine of GS to the sine of AS is as the ratio of the sine of GH to the sine of AH. The angle GEA is right, so the arc GE divides the angle DEH into two halves. Since the angle EDH is divided into two halves by the arc AD and the arc ZH comes out from the point of intersection to the angle DHE, then BH divides the angle EHD into two halves by the previous demonstration.

And as for him saying that the ratio of the sine of ET to the sine of TD is as the ratio of the sine of EK to the sine of KD since each of them is the product of the ratio of the sine of EG to the sine of GZ and the ratio of the sine of ZA to the sine of AD then we are used to marking the intersection of EG and DH as L. The ratio of the sine of ET to the sine of TD is also the product of the ratio of the sine of EG to the sine of LG and the ratio of the sine of HL to the sine of HD, and the ratio of the sine of EK to the sine of KD is the product of the ratio of the sine of EZ to the sine of ZL and the ratio of the sine of HL to the sine of HD. Since angle EDH is divided by arc AD into two halves and the angle ADG is right, then the ratio of the sine of GE to the sine of GL is as the ratio of the sine of EZ to the sine of ZL. The ratio of the sine of HL to the sine of HD is shared so the ratio of the sine of EK to the sine of KD is as the ratio of the sine of ET to the sine of TD. Since this is so and the angle DHE is divided into two halves by the arc BH then the angle BHG is right by the proof presented, so because the angle TDG is equal to the angle GDH, the ratio of the sine of TD to the sine of DH is as the ratio of the sine of TG to the sine of GH. The angle DEH is already divided by the arc EG into two halves, so the ratio of the sine of TE to the sine of EH is also as the ratio of the sine of TG to the sine of GH, so the ratio of the sine of ET to the sine of EH is as the ratio of the sine of TD to the sine of DH. If we switch then the ratio of the sine of ET to the sine of TD is as the ratio of the sine of EH to the sine of DH. The angle DHE is already divided into the halves by the arc KH, so we see that the ratio of the sine of EK to the sine of KD is as the ratio of the sine of ET to the sine of TD. So the angle BHG is right, by the eleventh demonstration.

The Fourteenth Demonstration: If the greater leg of a triangle is not greater than a quarter-circle, and if we cut off two arcs from it and draw arcs from their endpoints to the base that enclose with the base angles that are equal to the angle that is enclosed with the other legs, then if the two arcs we cut off are equal, the differences between the pairs of arcs drawn from the endpoints are not equal. The lesser difference of the two is the difference between the two arcs that are closer to the lesser leg. If the differences between the pairs of drawn arcs are equal, then the two arcs that are cut off are not equal and the greater is the one adjacent to the apex of the triangle. If the sum of one of the cutoff arcs with the difference between the two arcs drawn from its endpoints is equal to the sum of the other cutoff arc with the difference between the two arcs drawn from its endpoints, then the two cutoff arcs are not equal and the greater of them is the one adjacent to the apex. If the difference between one of the cutoff arcs and the difference between the two arcs drawn from its endpoints is equal to the difference between the other cutoff arc and the difference between the two arcs drawn from its endpoints, then the cutoff arc that is adjacent to the apex is less than the other. In all cases, the ratio of the cutoff arc that is closer to the apex to the other cutoff arc is greater than the ratio of the difference between the arcs drawn from the endpoints of the closer arc to the difference between the arcs drawn from the endpoints of the farther arc.

Let the triangle ABG be given, and let the side BG be longer than the side AG such that the side BG is not greater than a quarter-circle. Let's cut off the two arcs GD and ZT from the arc BG and draw from their endpoints the arcs DE, ZH, TK such that they make with the base angles that are equal to the angle at A. Then I say that if the arc GD is equal to the arc ZT then the difference between the two arcs GA, ED is less than the difference between the two arcs ZH, TK; and if the differences we mentioned are equal, then the arc GD is greater than the arc ZT; and if the sum of the arc GD and the difference between the two arcs AG, DE is equal to the sum of the arc ZT and the difference between the two arcs ZH, TK, then the arc GD is greater than the arc ZT; and if the difference between the arc GD and the difference between the two arcs AG, DE is equal to the difference between the arc ZT and the difference between the two arcs ZH, TK, then the arc GD is less than the arc ZT.

And I say in all cases that the ratio of GD to ZT is greater than the ratio of the difference between the two arcs AG, DE to the difference between the two arcs ZH, TK.

Since the two figures ABG and EBD are triangles and they share the angle at B, and the angles at the points A, E are equal, then the ratio of the sine of BG to the sine of BD is as the ratio of the sine of GA to the sine of DE. In the same way, also, the ratio of the sine of DB to the sine of BZ is as the ratio of the sine of DE to the sine of ZH, and the ratio of the sine of ZB to the sine of BT is as the ratio of the sine of ZH to the sine of TK. The arc BG is greater than the arc GA and BG is not greater than a quarter-circle, so because this is so then from what we said we can expose what we mentioned earlier. We have shown these things and all of the things similar to them in the first treatise of the astronomy demonstrations.

It can be extracted from this demonstration, in regards to astronomy, that of the arcs of the ecliptic in one quarter, the ratio of whichever one is close to the point of equinox, to the farther is less than the ratio of the ascension of the closer arc to the ascension of the farther. We are already mentioning only one side of this. Now, as for the case where the angle of A is right, let AG be the same as BG from the ecliptic, and let the ratio of the sine of angle B from this second drawing to the sine of the right angle—i.e., the whole sine—be as the ratio of the sine of angle B from the first drawing to the sine of angle A there. We make BG as it was and draw to AB the perpendicular GA, we take GD, DZ, ZT as they were, and we draw the perpendiculars DE, ZH, and TK.

Since we make the ratio of the sine of angle B to the sine of the right angle as it was in the earlier drawing, and BG, BD, BZ, BT, AK, AH, AB, EA as they were; and the ratio of the sine of BG to the sine of GA is as the ratio of the sine of BD to the sine of DE, and as the ratio of the sine of BZ to the sine of ZH, and as the ratio of the sine of BT to the sine of KT; then these perpendiculars are equal to the corresponding drawn arcs. It is clear that if we extend them all then they would meet at a pole of AB, so let that be L. We draw all of the arcs to that point and draw the perpendicular GS from G to LD and the perpendicular TM from the point T to LH. Since LS is less than LG then SD is greater than the difference between LD and LG, which is the difference between AG and DE, so SD is greater than the difference between AG and DE. Since LM is less than LT then ZM is less than the difference between LT and LZ, which is the difference between ZH and TK, so ZM is less than the difference between ZH and TK. The angle GDS is greater than the angle TZM, so if GD is equal to ZT then we make the angle GDN equal to the angle TZM. If we draw on it the perpendicular GW then this will lie outside the triangle GDS because the angle S is right and therefore the angle SND is acute. Since GD is equal to ZT and the angle W is right and the angle GDW is equal to the angle TZM then DW is equal to ZM. DN is greater than SD and DW is greater than DN, and SD is greater than the difference between AG and DE. ZM is less than the difference between ZH and TK, so ZM is less than the sum of the difference between ZH and TK and the difference between AG and DE.

The general proof of all of the cutoff arcs being equal or not equal is like this: We repeat the arcs as they were and draw, with L as a pole and by the distance LD, the circle MDS so that it cuts off SZ [from LZ equal to] the difference between LZ and LD and cuts off MG [from LG equal to] the difference between LD and LG. We draw on the two points M, S a great circle, and it is clear that since LS and LM are equal and their sum is less than a semicircle then the angles LSM and LMS are equal and both acute, and therefore each arc coming out from L to the base MS—which is from the great circle—is less than each of the two arcs LS and LM. So the great-circle arc MS intersects the circle LD between the two points L and D. In the same way it intersects DG between the two points D and G, so let them intersect at E. Since the sides LS and LM are equal then the two angles LSM and LMS are equal, so the sines of the two angles EMG and ESZ are equal, so the ratio of the sine of EG to the sine of MG is as the ratio of the sine of EZ to the sine of ZS. So the ratio of the sine of DG to the sine of GM is greater than the ratio of the sine of ZD to the sine of ZS.

If DG is not less than ZD then the ratio of the sine of DG to the sine of GM is greater than the ratio of the sine of ZD to the sine of ZS. If DG is less than ZD then let's cut off DH from ZD to be equal to DG so that ZH is still not greater than DG. The ratio of DG to the difference between LD and LG is greater than the ratio of DH to the difference between LH and LD, and this is greater than the ratio of ZH—which is not greater than GD—to the difference between LZ and LH, so the ratio of DG to the difference between LD and LG is greater than the ratio of all of ZD to all of the difference between LZ and LD. In the same way we can see that the ratio of ZD to the difference between LZ and LD is greater than the ratio of ZT to the difference between LZ and LT, so the ratio of GD to the difference between LD and LG is much greater than the ratio of ZT to the difference between LT and LZ.

The Fifteenth Demonstration: If one of the two base angles of a triangle is acute and the other angle is right and the side subtending the right angle is not greater than a quarter-circle, and if we cut off two arcs from this side and draw from their endpoints perpendiculars to the base, then if the two arcs cut off are equal, the two arcs that lie between the perpendiculars are not equal. The greater of them is the one adjacent to the right angle, and in this is exposed all the things that we mentioned in the same way that we said earlier.

Let the triangle ABG be given and let its angle at the point B be acute and its angle at the point A be right, such that the side BG is not greater than a quarter-circle. Let’s cut off from the arc BG the two arcs GD, ZT and draw from their ends to the base AB the perpendiculars DE, ZH, TK.

Then I say that if the arc GD is equal to the arc ZT then the arc AE is greater than the arc HK, and if the arc AE is equal to the arc HK then the arc GD is less than the arc ZT, and if the sum of the two arcs AE, DG is equal to the sum of the two arcs HK, ZT then the arc GD is less than the arc ZT; and if the difference between the two arcs AG and DE is equal to the difference between the two arcs ZH and TK that are picked in the same way, then the arc GD is greater than the arc ZT, and in all cases, the ratio of AE to HK is greater than the ratio of GD to ZT.

Since BG is less than a quarter-circle and the angle at the point B is acute, and each of the angles at A and E is right, then the ratio of the sine of the sum of AB and BG to the sine of their difference is as the ratio of the sine of the sum of EB and BD to the sine of their difference, and is as the ratio of the sine of the sum of BH and BZ to the sine of their difference. Because of this, all of what we mentioned earlier is exposed. Also, if the arc BG is a quarter-circle and the arc AB is equal to it then all of what we mentioned gets exposed. We already showed these things in the first treatise of the astronomy demonstrations.

To show what we mean, we present with what we are saying: two introductions that are easy and beneficial.

Let triangle ABG be such that the angle B is acute and the angle A is right.

I say that any two arcs coming out from a pole of AB to AB cut off between them an arc from the arc BG such that ratio of its sine to the sine of the arc that the two arcs cut off between them from AB is as the ratio of the sine of either of the two angles that were made by two arcs coming from the pole of AB intersecting BG to the sine of the arc between BG and the pole of AB that subtends that angle. So let BG be not greater than a quarter-circle, let Z be the pole of AB, and let's draw from it to AB the two arcs ZDH, ZET, however they agree.

Then I say that the ratio of the sine of TH to the sine of ED is as the ratio of the sine of angle D to the sine of ZE, and also as the ratio of the sine of angle E to the sine of ZD.

If we draw the perpendicular EL from the point E to the arc ZH then the ratio of the sine of TH to the sine of EL is as the ratio of the sine of ZH—the whole sine—to the sine of ZE. The ratio of the sine of EL to the sine of DE is as the sine of angle D to the sine of the right angle L, so the ratio of the sine of TH to the sine of ED—the product of these two ratios—is as the product of the ratio of the sine of ZH the third to the sine of angle L the sixth and the ratio of the sine of angle D to the sine of ZE. The sine of ZH is equal to the sine of angle L, so the ratio of the sine of TH to the sine of DE is as the ratio of the sine of angle D to the sine of ZE and is as the ratio of the sine of angle E to the sine of ZD.

After that we repeat the triangle ABG as it was, and we draw the arc ZDH in such a way that the acute angle at D evaluates the arc DZ.

Then I say that any two arcs coming out from a pole of AB to the arc BH cut off between them an arc from the arc BD that is greater than the arc from BH that they cut off between them, and that any two arcs coming out from a pole of AB to the arc AH cut off between them an arc from the arc DG that is less than the arc from AH that they cut off between them.

If we draw the two arcs ZET, ZLS such that the arc ZDH is between them, then it is clear that the ratio of the sine of TH to the sine of ED is as the ratio of the sine of angle D to the sine of EZ. EZ is greater than the measure of angle D, so the sine of TH is less than the sine of DE. Also, since the ratio of the sine of HS to the sine of DL is as the ratio of the sine of angle D to the sine of LZ, and LZ is less than the measure of angle D, then the sine of HS is greater than the sine of DL. So the arc TH is less than the arc DE and the arc HS is greater than the arc DL. By this we see, in this improved figure, that it goes the same way if the arc ZDH is neither of the two arcs cutting off arcs from BD and BH and neither of the two arcs cutting off arcs from GD and AH, because whichever of the [acute] angles formed by intersecting these arcs with BG is closer to B is smaller and whichever of the arcs lying between the pole and BG is closer to B is greater; and conversely, whichever of these angles is closer to G, as long as its distance from B is not greater than a quarter-circle, is greater.

From that it is also clear that arc BD is equal to arc AH and BH is equal to GD, because the ratio of the sine of BD to the sine of BH is as the ratio of the sine of the right angle H to the sine of angle D, and the ratio of the sine of AH to the sine of GD is as the ratio of the sine of HZ to the sine of ZD, and the sine of HZ is equal to the sine of the right angle H and the sine of ZD is equal to the sine of angle D.

If we present this, then the sum of AB and BG in Menelaus' figure is not greater than a quarter-circle. Let's make angle B from triangle ABG with the difference between AB and BG in the earlier figure as its measure, and let BG be a quarter-circle and draw the perpendicular AG to be equal to the difference between AB and BG in the earlier figure and draw GD to be equal to the sum of GD and AE there, and ZD to be equal to the sum of ZD and EH there, and ZT to be equal to the sum of ZT and

KH there, and we draw the perpendiculars DE, ZH, and TK so that DE here is the difference between BD and BE in the earlier figure, ZH is the difference between BZ and BH in the earlier figure, and TK is the difference between BT and BK there. The ratio of GD to ZT is greater than the ratio of the difference between AG and DE to the difference between ZH and TK, so the ratio of the sum of GD and AE in the earlier figure to the sum of ZT and KH is greater than the ratio of the difference between GD and AE—which in this figure is the difference between AG and DE—to the difference between ZT and KH there—which is here the difference between ZH and TK.

Since this is like this whenever the sum of AB and BG is not greater than a quarter-circle, then for the difference between GD, ZT and AE, KH the ratio of GD, as Menelaus said, to ZT is less than the ratio of AE to KH.

If the sum of BT and BK is not less than a quarter-circle, then it is the ratio of the sum of ZT and KH to the sum of GD and AE. This can be easily shown from completing each one of AB and BG to a semicircle in the earlier figure, and in the same way for the difference between KH, AE and ZT, GD, the ratio of GD to ZT is less than the ratio of AE to KH.

The Sixteenth Demonstration: We can show that from a different view.

We extend the arcs AG, ED, HZ, and KT to the pole of AB, which is L. Then the ratio of the sine of AB to the sine of BE is as the product of the ratio of the sine of GA to the sine of DE—which is as the ratio of the sine of BG to the sine of BD—and the ratio of the sine of LD to the sine of LG, and therefore the ratio of the sine of AB to the sine of BE is greater than the ratio of the sine of GB to the sine of BD. We also see that the ratio of the sine of BE to the sine of BH is greater than the ratio of the sine of BD to the sine of BZ, and that the ratio of the sine of BE to the sine of BK is greater than the sine of BD to the sine of BT. Therefore, the ratio of the sine of EK to the sine of AK is less than the ratio of the sine of DT to the sine of TG, so the ratio of the sine of AK to the sine of EK is greater than the ratio of the sine of TG to the sine of TD. In the same way we also see that the ratio of the sine of EK to the sine of KH is greater than the ratio of the sine of TD to the sine of TZ. Also, since the ratio of the sine of BH to the sine of BK is greater than the ratio of the sine of BZ to the sine of BT, the ratio of the sine of AK to the sine of AH is less than the ratio of the sine of TG to the sine of GZ; and therefore the ratio of the sine of AH to the sine of AE is less than the ratio of the sine of GZ to the sine of GD. Since these things are as we described then all of what we said earlier is exposed, so the ratio of AE to KH is greater than the ratio of GD to ZT.

In this improvement we see it in another way that is easier and clearer. Since the ratio of the sine of GD to the sine of AE—which evaluates the angle GLD—is as the ratio of the sine of LG to the sine of angle LDZ, and LZ is greater than LG, the ratio of the sine of GD to the sine of AE is less than the ratio of the sine of ZD to the sine of EH. In the same way we see that the ratio of the sine of ZT to the sine of KH is greater than the ratio of the sine of ZD to the sine of EH, so the ratio of the sine of GD to the sine of AE is much less than the ratio of the sine of ZT to the sine of KH. If GD is not greater than ZD and AE is less than GD then the ratio of GD to AE is less than the ratio of ZD to EH. If GD is greater than ZD then we make DN equal to ZD so that NG is not greater than ZD, and we draw the arc LNS. From what we mentioned we see that the ratio of the arc DN to the arc ES is less than the ratio of the arc ZD to the arc EH, and that the ratio of the arc NG to the arc AS is less than the ratio of the arc DZ to all of the arc EH, so the ratio of all of the arc GD to all of the arc AE is less than the ratio of the arc DZ to arc EH.

If the arc EH is greater than the arc ZD and GD is not less than ZD, and the ratio of the sine of GD to the sine of AE is less than the ratio of the sine of ZD to the sine of EH, and DE is greater than GD, then the ratio of the arc GD to the arc SE is less than the ratio of the arc ZD to the arc EH. If the arc ZD is greater than the arc GD then we make the arc DM to be equal to the arc GD so that MZ is not greater than GD, and we draw the arc LMO. We see from what we mentioned that the ratio of the arc GD to the arc AE is less than the ratio of the arc DM to the arc EO and less than the ratio of the arc MZ to the arc HO, so the ratio of the arc GD to the arc AE is less than the ratio of all of the arc ZD to all of the arc EH. In the same way we make DZ and ZT different quantities, each of them, until we see that the ratio of the arc ZD to the arc EH is less than the ratio of the arc ZT to the arc KH, so we see that the ratio of the arc AH to the arc KH is greater than the ratio of the arc GZ to the arc ZT.

What we can get from this as regards astronomy is that the ratio of an arc of the ecliptic that is closer to the equinox to its ascension in the horizon is greater than the ratio of an arc farther from the equinox to its ascension.

It's like this: how Menelaus showed that the ratio of the sine of ZG to the sine of GD is greater than the ratio of the sine of AH to the sine of AE, and that the ratio of the sine of DT to the sine of ZT is less than the ratio of the sine of EK to the sine of HK, has a perspective that is easier and better. As was shown earlier, the ratio of the sine of AH to the sine of ZG is as the ratio of the sine of angle Z to the sine of LG and the ratio of the sine of AE to the sine of GD is as the ratio of the sine of angle D to the sine LG. The acute angle at D is greater than the acute angle at Z, so the ratio of the sine of AH to the sine of ZG is less than the ratio of the sine of AE to the sine of GD, so the ratio of the sine of ZG to the sine of GD is greater than the ratio of the sine of AH to the sine of AE. Similarly, the ratio of the sine of EK to the sine of DT is as the ratio of the sine of angle D to the sine of LT and the ratio of the sine of KH to the sine of ZT is as the ratio of the sine of angle Z to the sine of LT, and the acute angle at Z is less than the acute angle at D, so the ratio of the sine of KH to the sine of ZT is less than the ratio of the sine of EK to the sine of DT, so the ratio of the sine of DT to the sine of ZT is less than the ratio of the sine of EK to the sine of KH.

The Seventeenth Demonstration: If the legs of a triangle are not equal and its greater leg is not greater than a quarter-circle, and if we cut off two arcs from the lesser leg and draw arcs from their endpoints to the base that enclose with the base angles that are equal to the angle that is enclosed with the greater leg, and draw other arcs perpendicular to the base, then of the base if the two arcs between the arcs making the equal angles are equal, the two arcs between the perpendiculars are not equal: The greater of them is the one near the lesser leg; and if the two arcs between the perpendiculars are equal, the two arcs between the arcs making equal angles are not equal: The greater of them is the one near the greater leg, and what occurs from this can be exposed as we mentioned.

Let the triangle ABG be given and let AG be greater than BG, where AG is not greater than a quarter-circle, and let's cut off from BG the two arcs GD and DZ and draw from their endpoints to the base AB the two arcs DE, ZH that enclose with it angles equal to the corresponding angle at A, and let the arcs GT, DK, ZL be perpendicular to AB. Then I say that if the arc AE is equal to the arc EH, the arc TK is less than the arc LK, and if the arc TK is equal to the arc LK, the arc AE is greater than the arc EH. In all cases, the ratio of AE to EH is greater than the ratio of TK to LK.

In the two triangles ABG and EDB, the angles at the points A and E are equal and the angle at B is shared, and we have drawn in these triangles the perpendiculars GT and DK, so the ratio of the sine of AT to the sine of TB is as the ratio of the sine of EK to the sine of KB—and in the same way, the ratio of the sine of EK to the sine of KB is as the ratio of the sine of HL to the sine of LB—by the fourth demonstration. If we switch these quantities we preserve proportionality. AT is greater than TB because AG is greater than BG, so if the arc KT is equal to the arc KL, then in the first picture, the difference between the two arcs AT and EK—which is the sum of the arcs AE and TK—is greater than the difference between EK and LH—which is the sum of EH and KL. In the second picture, the sum of the two arcs AE and TK is greater than the sum of the two arcs EH and KL. So in both of the pictures, the arc AE is greater than the arc EH. In the case where AE is equal to EH, in the first picture the sum of the arcs AE and TK, which is the difference between AT and EK, is less than the sum of KL and EH, which is the difference between EK and LH. In the second picture, the sum of AE and TK is less than the sum of EH and KL. Therefore the arc TK is less than the arc KL. In all cases, the ratio of AE to EH is greater than the ratio of TK to KL, and we see from this and from our earlier work that the ratio of AE to EH is greater than the ratio of GD to DZ.

To show this, we make the ratio of the sine of angle BAG to the whole sine as the ratio of the sine of BT to the sine of AT, and we make [the new] AB equal to [the old] AT, and we draw [the new] BG perpendicular to AG. Because we made the ratio of the sine of angle BAG to the whole sine as the ratio of the sine of BT to the sine of AT, and made AB equal to AT, and the ratio of the sine of AB to the sine of BG is as the ratio of the right angle G to the sine of angle BAG, so the [new] arc BG is equal to the [old] arc BT. We make AD [from the new AB] equal to [the old] EK and AE [from the new AB] equal to [the old] HL, so [the new] BD is equal to the sum of AE and TK in the first pictures and DE is equal to the sum of EH and KL. BD is equal to the difference between AE and TK in the second picture, and DE is first pictures and DE is equal to the sum of EH and KL. BD is equal to the difference between AE and TK in the second picture, and DE is the difference between EH and KL. The ratio of BD to the difference between BG and DZ, as shown earlier, is greater than the ratio of DE to the difference between DZ and EH, so the ratio of the sum of [the old] AE and TK to TK is greater than the ratio of the sum of EH and KL to KL. If we subtract then the ratio of AE to TK is greater than the ratio of EH to KL. In the second picture, the ratio of the difference between AE and KL to KL is greater than the ratio of the difference between EH and TK to the TK, so if we add then the ratio of AE to TK is greater than the ratio of EH to KL. If we switch then the ratio of AE to EH is greater than the ratio of TK to KL.

What appears from this in regards to astronomy is that the ratio of the ascension of the arc to the solstice in the oblique sphere to the ascension of the arc to the point of equinox there is greater than the ratio of their right ascensions. If we were to imagine AG as part of the ecliptic in the earlier and AB as part of the equator, and GB as some horizon inclined on the equator, and G as being at either the beginning of Capricorn or the end of Gemini in the first picture and either the beginning of Cancer on the end of Sagittarius in the second picture, then the arc AB would be the ascension of AG. In the case where G is at the beginning of Capricorn, the point A is in the beginning of Libra below the Earth, and in the case where G is at the end of Gemini, A is at the beginning of Aries above the Earth. AT is the ascension of AG in the upright sphere and BT is its equivalence in the horizon BG, EB is the ascension of DE and BK is its equivalence, and HB is the ascension of ZH and BL is its equivalence; so it follows that AE is the ascension between the two arcs AG and DE, TK is its equivalence, EH is the ascension between DE and ZH, and LK is its equivalence. So it has been shown that the ratio of AE to EH is greater than the ratio of TK to KL.

The Eighteenth Demonstration: We can also see that if the angle at A of the triangle ABG is obtuse and the one at B is acute and the arc BG is not greater than a quarter-circle, and if we cut off the arcs GD and DZ from BG and draw the arcs DE and ZH to the base AB so that they make angles with it that are equal to the corresponding angle at A, and we draw the perpendiculars GT, DK, and ZL, then what we mentioned can be exposed. In all cases, the ratio of AE to EH is greater than the ratio of GD to DZ.

We can also see, from what we mentioned, that since we made AB equal to AT and made the ratio of the sine of angle BAG to the whole sine as the ratio of the sine of BT to the sine of AT in the earlier demonstration, then here we imagine that the ratio of the sine of angle BAG to the whole sine is as the ratio of the sine of AT to the sine of BT, since BT is greater than AT here. If we go the same way as before, we see from this that the ratio of TK to KL is greater than the ratio of the difference between AE and TK to the difference between EH and KL. Since the angle T of the triangle AGT is right and the acute angle A is equal to the acute angle E from the triangle EDK that has a right angle, then the base EK is less than the base AT, so AE is less than TK. In the same way, EH is less than KL. The ratio of the excess of TK over AE to the excess of KL over EH is less than the ratio of TK to KL, so the ratio of AE the remaining to EH the remaining is greater than the ratio of TK to KL.

It appears from this that of the arcs which are near a solstice in the middle of the ecliptic, the ones from the beginning of Capricorn to the end of Gemini are such that the ratio of one's ascension in the oblique sphere to its ascension in the upright sphere is greater than this ratio for an arc closer to a point of equinox.

The Nineteenth Demonstration: If the legs of a triangle are not equal and its greater leg is not greater than a quarter-circle, and if we draw from the apex point to the base an arc inside the triangle that is not less than the lesser of the two legs and cut off two [adjacent] arcs from the lesser of the two legs and draw arcs from their endpoints to the base that enclose with it angles equal to the angle enclosed by the greater leg, and if other arcs enclose with the base angles equal to the angle enclosed with the arc that was first drawn: then the sum gets exposed, that we mentioned in the earlier demonstrations, and in all cases the ratios of the arcs lying between the arcs enclosing with the base angles equal to the angle enclosed with the greater leg, one to another, are greater than the ratios of the arcs lying between the other drawn arcs, if we take the numerator one in each ratio to be the arc closer to the greater leg and denominator one to be the arc that replaces it.

Let the triangle ABG be given and let AG be greater than BG such that AG is not greater than a quarter-circle, and let's draw from the point G to the base AB the arc GD that is not less than the arc BG. Let's cut off from BG the two arcs GE and EZ and draw from their endpoints to the base AB the arcs EH and ZT where they enclose with AB angles equal to the angle that is at A. Let's also draw the two arcs EK and ZL where they enclose with AB angles equal to the angle GDB. Then I say that the ratio of AH to HT is greater than the ratio of DK to KL.

If the angle at B is right then the ratio of the sine of AB to the sine of BH is as the ratio of the sine of BD to the sine of BK, and the ratio of the sine of BH to the sine of BT is as the ratio of the sine of BK to the sine of BL, by the fourth demonstration; so we see that what we mentioned follows.

This is clear when we set measure the angle A from BD and made AB subtend the right angle, I mean we drop a perpendicular with distance AB from the angle to the next arcs which enclose that angle with the earlier arcs, such as DB. Also, with distance BH such as BK and distance BT such as BL.

It appears from this, in regards to astronomy, that for an arc near a solstice, the ratio of its ascension in the upright sphere to the ascension in the upright sphere of an arc closer to the sphere is greater than the ratio of the equivalence of the first arc to the equivalence of the other arc.

The Twentieth Demonstration: And also, we make the angle that is at B right and draw to the base AB the perpendiculars GM, EN, and ZS. Since GD is not less than GB then DM is not less than MB. We see as we showed before that the ratio of the sine of AM to the sine of MB is as the ratio of the sine of HN to the sine of NB and is as the ratio of the sine of TS to the sine of SB, and the ratio of the sine of DM to the sine of MB is as the ratio of the sine of KN to the sine of NB and is as the ratio of the sine of LS to the sine of SB—all of this is by the fourth demonstration—. But the arc AM is greater than the arc MB and the arc DM is not less than the arc MB, and each of the two arcs AG, AM is not greater than a quarter-circle, so the ratio of the difference between the two arcs AB, BH to the difference between the two arcs BH, BT is greater than the ratio of the difference between the two arcs DB, BK to the difference between the two arcs KB, BL.

We also see that since the ratio of AD to DB is greater than the ratio of HK to KB then this ratio is greater than the ratio of TL to LB.

Since these sines are in the proportions we have shown, then the ratio of the sine of AM to the sine of DM is as the ratio of the sine of HN to the sine of KN and is as the ratio of the sine of TS to the sine of LS, and if we switch then the ratio of the sine of AM to the sine of HN is as the ratio of the sine of DM to the sine of KN and the ratio of the sine of HN to the sine of TS is as the ratio of the sine of KN to the sine of LS. By the fourteenth demonstration, the ratio of the difference between AM and HN—which is the sum of AH and MN—to the sum of HT and NS—which is the difference between HN and TS—is greater than the ratio of the difference between DM and KN—which is the sum of MN and DK—to the sum of KL and NS—which is the difference between KN and LS—and therefore the ratio of AH which is the sum of the difference with MN, to HT which is the sum of the difference with NS, is greater than DK—which is the sum of the other difference which has a lesser ratio to the one after it with MN—to KL, which is the sum of the difference after that one with NS.

The Twenty-First Demonstration: We also see that if the angle at A in the triangle ABG is acute and the angle B is obtuse and the side AG is not greater than a quarter-circle; and if we draw from the point G to the base AB the arc GD, and cut off from AG the two arcs GE and EZ, and draw the two arcs EH and ZT to make angles with the base AB equal to the angle at B, and draw the two arcs EK and ZL to make angles with the base AB equal to the angle at E; then the ratio of DL to KL is greater than the ratio of BH to HT.

We draw the perpendiculars GM, EN, and ZS. Then the ratio of the sine of AM to the sine of MB is as the ratio of the sine of AN to the sine of NH, and is as the ratio of the sine of AS to the sine of ST. The ratio of the sine of AM to the sine of MD is as the ratio of the sine of AN to the sine of BD and is as the ratio of the sine of AS to the sine of SL. Therefore, the ratio of the difference between the arcs GA and AK to the difference between KA and AL is greater than the ratio of the difference between BA and AH to the difference between HA and AT.

The conclusion of this with regards to astronomy is that, of the arcs in the middle beginning of Capricorn and the end of Gemini, the ratio of the ascension of an arc closer to a solstice to the ascension of one farther from it is greater when the inclination of the horizon is greater. The reverse of this is in the other side.

The Twenty-Second Demonstration: If there are two great circles in the surface of a sphere such that each one of them is inclined on the other, and two points on one of them are picked to be not on opposite ends of a diameter and if we draw from them two perpendiculars to the other circle: then the ratio of the sine of the arc between the feet of the two perpendiculars to the sine of the arc that is between the two points that we picked is as the ratio of the rectangle enclosed by the diameter of the sphere and the diameter of a circle that is tangent to one of the two [great] circles and parallel to the other, to the rectangle that is enclosed by the diameters of the two circles that go through the two points that we picked on one of the two great circles and are parallel to the other.

Let the two great circles AB, BG be given and let each of them be inclined on the other, and let's pick on AB the two points D and E and draw from them to the circle BG the two perpendiculars DG and EH. Then I say that the ratio of the sine of GH to the sine of DE is as the ratio of the rectangle enclosed by a diameter of the great circle and by a diameter of the circle that is tangent to AB and parallel to BG; to the rectangle that is enclosed by diameters of the circles that go through the two points D and E and parallel to the circle BG.

We take out the two arcs GD and HE to a pole of the circle BG, which is the point Z, and we draw from the point Z to the circle AB the perpendicular ZA. Since each of the two angles ZAE and ZHB is right and the angle AEZ is equal to the angle BEH then the ratio of the sine of AZ to the sine of ZE is as the ratio of the sine of BH to the sine of BE. Because of what we drew in this picture, the ratio of the sine of GH to the sine of DE is the product of the ratio of the sine of GZ to the sine of DZ and the ratio of the sine of BH to the sine of BE, which we have shown is as the ratio of the sine of AZ to the sine of ZE. So the ratio of the sine of GH to the sine of DE is as the ratio of the rectangle that is enclosed by the sine of GZ and the sine of AZ to the rectangle that is enclosed by the sine of DZ and the sine of ZE. But the sine of GZ is the radius of the sphere and the sine of AZ is the radius of a circle that goes through A and is parallel to the circle BG and tangent to the circle AB. As for the sines of the two arcs DZ and ZE, these are the radii of the two circles that go through the two points D and E and are parallel to the circle BG.

As for him saying that the ratio of the sine of GH to the sine of DE is as the ratio of the rectangle enclosed by the diameter of the sphere and the diameter of the circle that is tangent to AB and parallel to BG, to the rectangle enclosed by the diameter of the two circles that are parallel to the circle BG and go through the two points D and E, we draw from the point E to GD the perpendicular EL. The ratio of the sine of GH to the sine of EL is as the ratio of the sine of HZ to the sine of EZ, and the ratio of the sine of EL to the sine of ED is as the ratio of the sine of AZ to the sine of AD, and the ratio of the sine of the angle Z to the sine of DE is as the ratio of the sine of angle D to the sine of ZE and is as the ratio of the sine of angle E to the sine of ZD. Whenever we want to pick a point from arc AZ for a circle parallel to circle BG to go through, the ratio of its diameter to the diameter of the circle parallel to BG that goes through D is as the ratio of the sine of GH to the sine of DE, for any point this circle parallel to the circle BG goes through. So the ratio of its diameter to the diameter of the circle parallel to circle BG that goes through E is as the ratio of the sine of GH to the sine of DE.

We take AB as the measure of BH and TK as the measure of GH, and however the points T and K fall, we take out the arcs ZTM and ZKL to BG. Then the ratio of the sine of ZT to the sine of ZD and the ratio of the sine of ZK to the sine of ZE is as the ratio of the sine of GH to the sine of DE; so the acute angle at D is measured from completing the inclined of BG and the acute angle at E is measured from completing the inclined of BH. Theodosius has proved this demonstration but we will introduce an introduction to his demonstration.

Let triangle ABG be such that its side BG is not less than side AG, and however it happens, the line BE comes out to the side AG. Then I say that the ratio of GA to AE is greater than the ratio of angle BEA to angle BGA.

We draw GZ parallel to BE and extend AB to it until they meet of Z. Since BG is not less than AG then ZG is greater than BG. Making G a center, we draw around it by the distance of GB the arc ABH, so the arc BH meets the line ZG between Z and G. Since [the line] AG is not less than [the arc] BH then they meet at A. We see that the triangle ABG is less than the sector ABG and the triangle ZBG is greater than the sector BHG. But the ratio of triangle ZBG to triangle ABG is as the ratio of ZB to BA, so the ratio of ZB to BA is greater than the ratio of sector BHG to sector AGB. If we add then the ratio of ZA to AB is greater than the ratio of sector AGH to sector AGB, which is as the ratio of angle AGZ to angle AGB. Angle AGZ is equal to the angle AEB and the ratio of ZA to AB is as the ratio of GA to AE, so the ratio of GA to AE is greater than the ratio of angle AEB to angle AGB.

Let the circle ABGD be perpendicular to some parallel circles and let the circle AZG be the greatest of them, let the circle HZT be inclined on the parallel circles and perpendicular to the circle ABGD. Let E be the center of the sphere and let B and D be the two poles of the parallel circles, and let's draw the great-circle arc NC.

Then I say that the ratio of the diameter of the sphere to the diameter of the circle that is parallel to circle AZG and tangent to circle HT at the point H is greater than the ratio of arc AC to arc HN.

We draw the diameters AEG, BED, HET, and HLS such that HLS is parallel to the diameter AEG. As for HLS, it is the diameter of the circle tangent to circle HZT and parallel to circle AZG; and as for the diameter BED, it is perpendicular to all of the parallel diameters. We draw the circle KNM as one of the parallel circles and draw the diameter KFM. The section that is shared by the two circles HZT and KNM, which are perpendicular to the circle ABGD, is perpendicular to the plane of ABGD. Since B is a pole for the circles AZG and KNM then arc KN is related to arc AC. The point F, shared by diameter KFM and diameter BED which goes through the poles of the two circles AZG and KNM, is the center of circle KNM, so the angle NFO is evaluated by arc AC. The angle NEO is evaluated by arc HN. Since the angle at F is right then EO is greater than FO, so we make OJ equal to FO and connect FJ. Since the angle NOJ is right and OJ is equal to FO and NO is shared, than the angle OJN is equal to the angle OFN. We see that the ratio of line EO to line OJ is greater than the ratio of angle OJN to angle OEN, so the ratio of EO to OF is greater than the ratio of the angle AEC to the angle OEN, which is as the ratio of the arc AC to the arc HN. The ratio of EO to OF is as the ratio of the arc EH to the arc HL. So the ratio of EH, which is the radius of the sphere to HL, which is the radius of the circle that is tangent to circle HZT at H, is greater than the ratio of the arc AC to the arc HN.

The Twenty-Third Demonstration: Now let each of the two circles AB, BG be inclined on the other, and we draw a circle that goes through their poles and has the points Z, A, T on it, letting the point Z be a pole to the circle TB. We also draw from the point Z a great-circle arc with the points Z, K, M on it, such that the sine of ZK is the proportional mean between the sines of the two arcs ZM and ZA, and by doing that, we are making the diameter of the circle parallel to BT and going through the point K the proportional mean between the diameter of the sphere and the diameter of the circle tangent to the circle AB and parallel to the circle BT. Then I say that the difference between the two arcs KB and MB is determined, and it is greater than the difference between any two arcs drawn on this side. Since the ratio of the sine of MZ to the sine of ZK is as the ratio of the sine of ZK to the sine of ZA then the ratio of the sine of TM to the sine of AK is as the ratio of the sine of KB to the sine of MB. But the arc BT is equal to the arc BA, so the arc TM is equal to the arc KB and the arc KA is equal to the arc MB. Since the ratio of the sine of MZ to the sine of KZ is as the ratio of the sine of KZ to the sine of AZ, then the ratio of the diameter of the sphere to the diameter of the parallel circle that is tangent to the circle AB at A is as the ratio of the square generated from the sine of ZM to the square generated from the sine of ZK. This ratio is as the ratio of the square of the sine of TM to the square of the sine of KA, so if we add and flip the ratio then the ratio of the sum of the two diameters that we mentioned in the beginning to their difference is as the ratio of the square of the diameter of the sphere to the difference between the squares of the sines of TM and KA. The two diameters that we mentioned in the beginning are determined, so the difference between the two of them is determined, so the difference between the squares of the sines of the two arcs TM and KA is determined. Since the sum of these two arcs is a quarter-circle then their difference is determined, which is the same as the difference between the two arcs MB and BK.

Then I say that this is the greatest difference, of all differences between pairs of arcs drawn on this side.

We draw from the point Z the two arcs ZDG and ZEH. Then the ratio of the sine of GM to the sine of DK is as the rectangle enclosed by the diameter of the sphere and the diameter of the circle parallel to the circle BT and tangent to the circle AB at A, to the rectangle enclosed by the diameters of the two circles that go through K and D and are both parallel to the circle BT. The rectangle enclosed by the diameter of the sphere and the diameter of the circle parallel to the circle BT and tangent to the circle AB at A is equal to the square generated from the diameter of the circle going through K and parallel to circle BT, and this is greater than the rectangle enclosed by the diameters of the circles that go through K and D and are both parallel to circle BT, so the arc GM is greater than the arc DK. We also see that the arc MH is smaller than the arc KE, and because this is so, it is clear that the difference between the arc KB and the arc MB is greater than the difference between the arc EB and the arc BH, and is greater than the difference between the arc DB and the arc BG. That shows that the arc ZM is the arc that cuts off two arcs from circles AB and BT whose difference is greater than the difference between any two arcs cut off on this side.

As for saying that the arc HM is less than the arc EK and the arc MG is greater than the arc KD when such an arc ZKM is drawn, we show that by a less strict way. It is because the ratio of the sine of MG to the sine of KD is as the ratio of the sine of angle D to the sine of ZK, and angle D is greater than the arc ZK. Also, the ratio of the sine of MH to the sine of EK is as the ratio of the sine of angle E to the sine of ZK, and angle E is less than ZK. As for him mentioning knowledge of the difference between MT and AK, an easier [way] is the ratio of the sum of the diameter of the sphere and the diameter of the circle tangent to the circle AB at A to the sine of AZ is as the ratio of the sum of the squares of the sines of MT and AK to the square of the sine of AK. Also, what we said is that the ratio of the radius of the sphere to the sine of the difference between the two arcs MT and AK is as the ratio of the sum of squaring the chords of the arcs TZ and AZ to the square of the chord of the arc AT. Since the difference between MT and AK is the difference between KB and MB, and the sum of KB and MB is a quarter circle, then the sine of their sum is the radius of the sphere and the ratio of the sine of the sum of the two arcs KB and MB to the sine of their difference is as the ratio of the sine of half of completing the acute angle Z to two right angles to the sine of half of the arc which evaluates the acute angle B, multiplied by itself.

The Twenty-Fourth Demonstration: Now let the point Z be a pole of the circle BG such that the arc BD is not greater than a quarter-circle, and let the arc GH be greater than the arc DE. Then I say that the ratio of GH to DE is less than the ratio of the diameter of the sphere to the diameter of the circle that is parallel to circle BG and goes through D.

Since the arc BD is not greater than a quarter-circle and DE is less than GH and BE is greater than BH, and the ratio of the sine of GH to the sine of DE is the product of the ratio of the sine of GZ to the sine of ZD and the ratio of the sine of BH to the sine of BE, then the ratio of the sine of GH to the sine of DE is less than the ratio of the sine of the arc GZ to the sine of the arc ZD and this is the ratio of the diameter of the sphere to the diameter of the circle going through D and parallel to the circle BG. Therefore, the ratio of GH to DE is less than the ratio of the diameter of the sphere to the diameter of the circle going through D and parallel to the circle BG; because the arc ZG is a quarter-circle and the arc GH is less than a quarter-circle.

Also, if the ratio of the sine of GH to the sine of DE is as the ratio of the rectangle enclosed by the diameter of the sphere and the diameter of the circle parallel to the circle BG and tangent to the circle BD to the rectangle enclosed by the diameters of the two circles going through the two points D and E that are parallel to the circle BG, then I say that the ratio of GH to DE is greater than the ratio we mentioned.

Since the arc GH is greater than the arc DE then the ratio of the arc GH to the arc DE is greater than the ratio of the sine of the arc GH to the sine of the arc DE—which is the ratio of the rectangle enclosed by the diameter of the sphere and the diameter of the circle that is tangent to BD and parallel to BG to the rectangle that is enclosed by diameters of the two circles that go through the two points D and E and parallel to the circle BG.

So it is clear that when the arc GH is greater than the arc DE then a ratio of a greater to a smaller is greater and less than any arbitrary ratio.

As for him saying that the ratio of GH to DE is greater than the ratio of the rectangle enclosed by the diameter of the sphere and the diameter of the circle that is tangent to BD and parallel to BG to the rectangle that is enclosed by diameters of the two circles that go through the two points D and E and parallel to the circle BG is in what was just mentioned. And again, the ratio of the arc GH to the arc DE is greater than the ratio of the sine of the angle D to the sine of the arc ZE and also greater than the ratio of the sine of the angle E to the sine of the arc ZE.

When it is clear that the ratio of the sine of GZ to the sine of ZD is greater than the ratio of the sine of GH to the sine of DE, it is not clear that the ratio of the arc GH to the arc DE is greater than the ratio of the sine of GZ to the sine of ZD.

As for him saying "and this is the ratio of the diameter of the sphere to the diameter of the circle going through D and parallel to the circle BG" is an evidence of the need for the demonstration of Theodosius. Because in the case where each of the arcs BD and BG is a quarter-circle, he proved that the ratio of the arc GH to the arc DE is greater than the ratio of the sine of GZ to the sine of ZD. So with a little more that we can add, we can see that in the case where none of them is a quarter-circle, the ratio of the arc GH to the arc DE being greater than the ratio of the sine of GZ to the sine of ZD is not because of the ratio of the sine of GZ to the sine of ZD is greater than the ratio of the sine of GH to the sine of DE.

Let's repeat from the demonstration of Theodosius the circles ABGD, AZG, and HZT with the diameters BD, AG and HT [respectively]. Let each of the arcs ZH and ZA be less than a quarter-circle such that the circle HZT is inclined on the circle ABGD, AZG, and HZT with the diameters BD, AG, and HT [respectively]. Let each of the arcs ZH and ZA be less than a quarter-circle such that the circle HZT is inclined on the circle ABGD. As we did earlier, we take out the arc NC and draw HLS parallel to the diameter AEG. From the point N, we take out NQ perpendicular to the diameter HET. The circle HZT is inclined on the plane of the circle ABGD towards the side of the semicircle HAT, so each line coming out from the diameter HET, in the plane of the circle ABGD towards the side of the inclination, encloses an acute angle with NQ. So a perpendicular coming out from the point N to the plane of the circle ABGD going through the diameter HET will be to the side of A. Let such perpendicular be NO and draw the line KOFM parallel to the diameter AEG which makes it the diameter of the circle going through N and parallel to the circle AG. Also, we draw the line RQXV parallel to the diameter AEG, and we connect FN, EN [and QE, QF]. It is clear that our construction is heading towards the angle NFO. Since it is greater than the angle NEQ, the angle EQN is right as the angle FON is also right, and EN is greater than FN, than EQ is greater than FO. Let QJ be equal to FO and connect NJ. Since NO is perpendicular to the plane of ABGD then NQ is greater than NO. So the angle NJQ is greater than the angle NFO and the ratio of EQ to QJ is greater than the ratio of the angle QJN to the angle QEN. So the ratio of EQ to FO is much greater than the ratio of the angle NFO to the angle NEQ. We draw OQ; then it is perpendicular to the diameter HET of the circle being inclined on the circle ABGD. So the angle XQO is obtuse. So XQ is less than FO. The ratio of EQ to QX is as the ratio of EH to HL so the ratio of EH to HL is greater than the ratio of EQ to FO. The ratio of EQ to FO is greater than the ratio of the angle QJN—which is greater than the angle NFO—to the angle NEQ. So the ratio of EH to HL is much greater than the ratio of the angle NFO to the angle NEQ.

Thereby knowing that OQ is perpendicular to the diameter HET, we take out the line NU to the diameter HET however they agree and connect OU. Since the angle UQN is right then the square of NU equals the sum of the squares of NQ and QU. Also, NO is perpendicular to the plane of the circle ABG, so the angle NOU is right. So the square of NU equals the sum of the squares of NO and OU. But the square of NQ equals the sum of the squares of NO and OQ. After dropping the square of NO the shared, it remains that the square of OU is equal to the sum of the squares of OQ and QU. So the angle OQU is right.

As for XQ is less than FO, we take out a perpendicular from the point Q to FO. Then it falls between the two points F and O and cuts off from the line FO a line equal to the line XQ.

The Twenty-Fifth Demonstration: And also, we set the arc GH to be less than the arc DE. Then the rectangle enclosed by the diameter of the sphere and the diameter of the circle parallel to the circle BG and tangent to the circle BD is smaller than the rectangle enclosed by the diameters of the two circles going through the two points DA and E that are parallel to the circle BG; because their ratio is as the ratio of the sines of the two arcs that we mentioned.

Then I say that the ratio of the arc GH to the arc DE is greater than the ratio of the diameter of the circle parallel to the circle BG and tangent to the circle BD to the diameter of the circle going through E and parallel to the circle BG and less than the ratio of the rectangle enclosed by the diameter of the sphere and the diameter of the circle parallel to the circle BG and tangent to the circle BD to the rectangle enclosed by the diameters of the two circles going through the two points D and E that are parallel to the circle BG.

Since the rectangle enclosed by the diameters of the two circles going through the two points D and E that are parallel to the circle BG is greater than the rectangle enclosed by the diameter of the sphere and the diameter of the circle parallel to the circle BG and tangent to the circle BD, then from the point Z we take out the great circle arcs ZKM and ZLN in such a way that each of the rectangles, where one is enclosed by the diameters of the two circles going through the two points D and L that are parallel to the circle BG and the other is enclosed by the diameters of the two circles going through the two points E and K that are parallel to the circle BG, is equal to the rectangle enclosed by the diameter of the sphere and the diameter of the circle parallel to the circle BG and tangent to the circle BD. So the point L lies between the two points D and E and by the equality of the rectangles we mentioned, the arc NG is equal to the arc DL and the arc MH is equal to the arc KE. As shown in the straight lines, the arc LE is either equal to the arc MG or the arc NH. But the arc LE is greater than the arc NH so it is equal to the arc MG. So the arc DK is equal to the arc NH and so all of the arc GH is equal to all of the arc KL and the arc MN is equal to the arc DE. We showed earlier that the ratio of MN to KL is less than the ratio of the diameter of the sphere to the diameter of the circle going through K and parallel to the circle BG—which is as the ratio of the diameter of the circle going through E and parallel to the circle BG to the diameter of the circle parallel to the circle BG and tangent to the circle BD—. So the ratio of DE to GH is less than the ratio we mentioned. Therefore, the ratio of GH to DE is greater than the ratio of the diameter of the circle parallel to the circle BG and tangent to the circle BD to the diameter of the circle going through E and parallel to the circle BG.

Also, when the arc GH is less than the arc DE, the ratio of the arc GH to the arc DE is less than the ratio of the sine of GH to the sine of DE. So it is less than the ratio of the rectangle enclosed by the diameter of the sphere and the diameter of the circle parallel to the circle BG and tangent to the circle BD to the rectangle enclosed by the diameters of the two circles going through the two points D and E that are parallel to the circle BG. So we see that when the endpoint of the quarter of a circle is the point D, the ratio of GH to DE is: less than the ratio of the diameter of the sphere to the diameter of the circle parallel to the circle BG and tangent to the circle BD and greater than the ratio of the diameter of the sphere to the diameter of the circle going through E and parallel to the circle BG. When the endpoint of the quarter of a circle is between the two points D and E, as the point L; and the two arcs DL and LE are equal then the ratio of GH to DE is less than the ratio we mentioned and is also greater than the ratio we described. If the two arcs DL and LE are not equal then the ratio of GH to DE is also less than the ratio of the diameter of the sphere to the diameter of the circle parallel to the circle BG and tangent to the circle BD but it is greater than the ratio of the diameter of the sphere to the diameter of the circle going through either one of the two points D or E, whichever is farther from L, and parallel to the circle BG.

In general, we acknowledge that the conclusion of Menelaus for the point L to lie between the two points D and E is incorrect. The point L could only lie between the two points D and E when the circle falling between them cuts two segments off BD and BG so that the difference between them is greater than the difference between the pair of segments being cut off from BD and BG by the pair of arcs coming out from the pole Z. When that circle doesn't fall between the points D and E, the point L doesn't lie between D and E. When the mentioned circle falls between both of the points D and E, it will also fall between both of the points D and L.

As for him saying that we take out these circles without a condition being added, the proof is never correct. This is when he takes out the two circles from the point Z to the circle BD where their arcs that lie between Z and BD are equal and so the endpoint of the quarter of the circle is between the points D and K. With a circle being taken out from the point Z to the endpoint of the side of the circle BD which is a quarter from B where its arc and the arc of the other that lie between Z and the circle BD are equal to the arc ZK, the proof is right.

But for the circle ZKM when the rectangle enclosed by the diameters of the two circles going through the two points E and K that are parallel to the circle BG is equal to the rectangle enclosed by the diameter of the sphere and the diameter of the circle parallel to the circle BG and tangent to the circle BD such that the endpoint of the quarter of the circle is not between the two points D and K, the proof is not correct. As for the sines of the arcs MH and EK are equal is not because the arcs MH and EK are equal, it is because they both are a semicircle. So the condition which must have been added is that both of the arcs are taken out to the quarter which both of the points D and E lie, equally, whether the point L has fallen between the points D and E, or not.

And again, the difference between both of the arcs BD and BG is greater than the difference between couple of arcs according to this property. So from the point Z, there is no circle can be taken out to the quarter, where the points D and E lie, so that: a rectangle enclosed by diameters going through D and a point on the quarter D and E are on but that circle have gone through is equal to the rectangle enclosed by the diameter of the sphere and the diameter of the circle parallel to the circle BG and tangent to the circle BD. On the contrary, either of these rectangles is greater or smaller. The square of the diameter of the circle going through D and parallel to the circle BG is equal to the rectangle enclosed by the diameter of the sphere and the diameter of the circle parallel to the circle BG and tangent to the circle BD. So it is also not possible in any way for both of the circles to be taken out to the quarter with D and E according to the property mentioned. But when the difference between BD and BG is the greatest difference between a pair of arcs, it is sufficient for the proof that the circle ZKM is taken out to the quarter of which the points D and E are so that the rectangle enclosed by the diameter of the two circles going through the two points E and K that are parallel to BG is equal to the rectangle enclosed by the diameter of the sphere and the diameter of the circle parallel to the circle BG and tangent to the circle BD. With the only obligation for both of the cases to be investigated for a possibility of both circles to be taken out to the mentioned quarter, or else on exactly one circle can be taken out. Then the proof is correct.

In the accordance of our findings, we say that when the ratio of the sine of GH to the sine of DE, in the case of a lesser to a greater, is less than the ratio of the sine of angle D to the sine of ZE and also less than the ratio of the sine of angle E to the sine of ZD; then each one of the two ratios is the ratio of the sine of GH to the sine of DE. In the process towards what is presented, it is clear that by the arc of the circle that is taken out from the point Z going through a point of the circle BD, which falls between Z and the point, the angle at E is measured and by the segment where such a point the circle is going through and falling between it and the point Z, the angle at D is measured.

As for us to desire that the ratio of GH to DE is greater than some ratio, as Menelaus mentioned, without the addition of the mentioned condition and also without possibilities mentioned, we do as follows.

We complete each one of the arcs BD and BG to a quarter-circle by the arcs DS and GM, and cut off MN to be equal to BE and NK to be equal to ED. We take out the great circle arcs ZLN and ZTK, and also the great circle arc ZS. Extend ZS to meet the circle BG at the point M. Since the angle BLN evaluates the completion of the inclination of MN and MN is equal to BE then the measure of the angle BLN equals the arc ZE. In the same way, the measure of the angle BTK equals the arc ZD. The ratio of the sine of BN, which is equal to ES, to the sine of BL is as the ratio of the sine of the angle BLN to the sine of the right angle at N and the ratio of the sine of ES to the sine of MH is as the ratio of the sine of ZE—which evaluates the angle BLN—to the sine of ZH, so the arc MH is equal to the arc BL. It is left that the arc LS is equal to the arc BH. In the same way, the arc ZL evaluates the angle BEH; because this is the angle which evaluates the completion of the inclination of BH. In the same way it is clear that the arc ZT evaluates the angle BDG. It is then clear that the ratio of the sine of KN to the sine of LT is as the ratio of the sine of DE to the sine of HG. KN is equal to DE. So LT is equal to HG. So we complete the rest of the proof in the way Menelaus had mentioned. So it is clear, that the ratio of GH to DE—when it is the ratio of a lesser to a greater—is: greater than the ratio of the sine of ZS to the sine of ZE and less than the ratio of the two rectangles Menelaus mentioned, one to the other. Also, it is less than the ratio of the sine of angle D to the sine of ZE and less than the ratio of the sine of angle E to the sine of ZD. And of what Menelaus has left to say, it is clear that we proceed with that.

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